A Simple, Unified, Exactly Solved Framework for Ten Puzzles in Macro-Finance

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Abstract

This paper works out a simple unified framework for a series of puzzles in macro-finance. It builds on the Rietz-Barro view, that a small probability of large crises or disasters generates is what generates risk premia in asset markets. During a disaster, an asset's fundamental value will fall by a time-varying amount. This time-varying amount generates time-varying risk premia, hence volatile asset prices, and partial predictability of future asset returns. Using the recent technique of linearity-generating processes (Gabaix 2007), the model is very tractable, and all prices are in closed form. Hence, the paper presents a simple, exactly solved, frictionless benchmark for a series of questions on asset prices. It provides a way to think about the following puzzles: (i) equity premium puzzle (ii) risk-free rate-puzzle (iii) excess volatility puzzle (the fact that equity prices are so volatile) (iv) value-growth puzzle (stocks with high price-dividend ratios have low future returns) (v) upward sloping nominal yield curve (vi) Fama-Bliss findings that a higher slope of the yield curve predicts higher risk premia on bond returns (vii) corporate bond spread puzzle (the spread between corporate and government bond rates are higher than warranted by the U.S. historical experience) (viii) characteristics vs covariance puzzles (simple numbers such as the price-dividend ratio of stocks predict future returns better that covariances with economic factors) (ix) partial predictability of aggregate stock market returns by price/dividend and consumption/wealth ratios (x) high price of deep out-of-the-money puts. The "probability of disaster" can be interpreted literally, or could simply model varying risk, risk aversion, or investor sentiment in a particularly tractable way. (JEL: E43, E44, F31, G12, G15)

1 Introduction

This paper proposes a simple, unified closed-form model for a series of puzzles in macroeconomics and finance.[1] It uses the idea of Barro (2006), Rietz (1988) and that there is always a possibility that some large macroeconomic crisis (e.g. the Great Depression, a war, or a revolution) could happen, and this type of risk creates risk premia on stocks, bonds, and other financial assets.[2] In their models, the intensity of potential disasters is constant. In the present paper, during a disaster, different assets see their fundamental value fall at a time-varying rate. Hence, assets have time-varying risk premia and volatile asset prices. For instance, stock prices are high when stocks are not very "risky", in the sense that their cash-flows would not fall by much if a crisis happened next period. This perceived riskiness mean-reverts, which leads to an expected mean-reversion of the price-dividend ratio of stocks. Hence stocks are very volatile, and their price-dividend ratio mean-reverts. The same dynamics hold for bonds and exchange rates.

The advantage of that formulation is that it allows for a very tractable model of stock, bonds and exchange rates, in which all prices are in closed forms. Hence, the paper presents a simple, tractable, frictionless benchmark for a series of questions on asset prices. Namely, it offers a way to think about the following puzzles.

Stock market: Puzzles about the aggregates

1. Equity premium puzzle.

2. Risk-free rate puzzle. For this and the above puzzle, the paper simply imports from Barro (2006) and Rietz (1988).

3. Excess volatility puzzle: The fact that stock prices are more volatile than warranted by a model with a constant discount rate.

4. Aggregate return predictability: Future aggregate stock market returns are partly predicted by Price/Earnings and Price/Dividend ratios, and the Consumption/Aggregate wealth ratio.

5. Counter-cyclical equity premium

   Stock market: Puzzles about the cross-section of stocks

6. Value/Growth puzzle: Stocks with a high (resp. low) P/D ratio have lower (resp. high) future returns, even controlling for their covariance the aggregate stock market.

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[1] It focuses on stocks, bonds and puts, while a companion paper (Farhi and Gabaix 2007) extends the model to international macroeconomics, in particular exchange rates, and the forward premium puzzle.

[2] Weitzman (2006) presents an influential related view, that the possibility of rare disasters stems from the uncertainty about the true model of the world. Longstaff and Piazzesi (2004) calibrate a model with a constant probability of rare disasters.

7. Characteristics vs Covariances puzzles: In various data sets and subsamples, characteristics of stocks (e.g. the P/D ratio) predict future returns as well or better than covariances with economically-motivated factors.

8. Existence of "good beta" factor (Campbell and Vuolteenaho 2004).

   Nominal bond puzzles: Government debt

9. On average, the long term rates are higher than short-term rates, i.e. the yield curve slopes up.

10. Fama-Bliss, Campbell-Shiller, Cochrane Piazzesi facts: A higher slope of the yield curve predicts excess positive returns on long term bonds.

11. A high continued deficit, or a high debt/GDP ratio leads to higher slope of yield curve (controlling for future inflation), and higher real long term rates. This is because it predicts an increase in inflation if there is a disaster.

    Nominal bond puzzles: Corporate debt

12. Corporate bond spreads are higher than warranted by a simple risk-neutral model

13. Higher Debt/GDP ratio leads to lower Corporate bond spreads (Krishnamurthy and Vissing-Jorgensen 2007)

    Options

14. High price of deep out-of-the-money puts.

To get a feel for the economics of the model, first consider bonds. The model postulates that, if a disaster happens at t, inflation will increase by some amount j_t. If $j_t > 0$, long term bonds are risky, and command a risk premium (they do badly in bad states of the world). On the other hand, short term bonds bear very little risk, and have a very small risk premium. Hence, long term rates are higher than short term rates – the nominal yield curve slopes up. Next, suppose that the amount by which inflation will increase, itself varies. Then the slope of the yield curve will vary. We have the ingredients for an economic theory of the yield curve. The model formalizes this idea, which turns out to account for many stylized facts on bonds.

The same mechanism is at work for stocks. Suppose that, when a disaster happens, the value of the earnings of a stock falls by j_t. That possibility yields a risk premium. If j_t is variable, it yields a time-varying risk premium. This time-varying risk premium makes stock prices volatile, and also makes them partly predictable via measures such as the dividend-price ratio.

In general, different stocks will have different falls in earnings (j_t) during the disaster. Stock that are expected to fall a lot will be very risky, and will have a low price – they will be categorized as "value" stocks (their price will be low compared to earnings, dividends, or book value) and they will have high expected returns. Those expected to do better will have a high price, and low expected returns – they are "growth" stocks. That generates a theory of value and growth stocks.

As the prospective defaults intensities vary across time and assets, they will generate comovement in returns – even in periods when a disaster does not happen. Hence, we generate comovement and comovement "factors", in periods in which no disaster happens.

The above ideas are quite straightforward, and many of them have been already formulated in other contexts. The main virtue of the paper is to articulate them in a unified, compact, tractable framework. I constantly tried to keep the model as tractable and streamlined as possible. This is achieved by using the linearity-generating processes developed elsewhere (Gabaix 2007) to maintain tractability.

I now mention further antecedents and motivation for this unification project.

John Cochrane (1999) has emphasized the tantalizing empirical similarities in the patterns of "excess volatility" and return predictability in stocks, bonds and exchange rates. Take stocks: if the rate of returns on all stocks was constant, stocks with a high D/P ratio should have a low expected price growth, but empirically, they have a higher than average price growth (that's an instance of the value puzzle). The same holds for exchange rates – that's the uncovered interest rate parity puzzle, a.k.a. the forward premium puzzle. If the rate of return of investing in different currencies was equalized, currencies with high interest rates should depreciate (so that the full expected return – interest rate plus expected capital gain – is equalized across countries). But empirically high-interest rate currencies tend to appreciate, not depreciate. The same, finally holds for bonds. Suppose a naive theory in which bonds should be constant over time. When long term bonds have high yields, one should expect lower than average capital gains from holding those long bonds (again, so that the full expected return – interest rate plus expected capital gain – is constant over time. However, the historical experience goes the other way. When long bond yields are high, expected capital gains are high (Campbell and Shiller 1991), the opposite of the simplest riskless theory. Cochrane concludes that this triad of puzzles suggests that a common mechanism might be at work.

The present paper aims at realizing Cochrane's program, and at presenting a unified mechanism for stocks, bonds and exchange rates. The work on exchange rate is developed in a companion paper, Farhi and Gabaix (2007)

The model is presented as fully rational, but it could be interpreted as a behavioral model. The changing beliefs about the intensity of possible disasters are very close to what the behavioral literature calls "animal spirits." The model's structure gives a time-consistent way to think about

the impact of changing "sentiment" on prices, in the time-series and the cross-section.[3]

In terms of predictions, the model behaves similarly to models with time-varying risk-aversion, which are chiefly done with external habit formation (Abel 1990, Campbell Cochrane 1999, Menzly Santos Veronesi 2004). The present proposal has two virtues, emphasized by Barro (2006). Because it conserves the usual i.i.d. structure with iso-elastic preferences, it is very tractable, and it meshes well with modern macroeconomic models, virtually all of which use those preferences.

The model is complementary to the literature on long term risk – which view the risk of assets as the risk of covariance with long-run consumption: e.g. Bansal and Yaron (2004), Bekaert et al. (2005), Croce, Lettau and Ludvigson (2006), Gabaix and Laibson (2002), Hansen, Heaton and Li (2005), Hansen and Scheinkman (2006), Julliard and Parker (2004), Lettau and Wachter (2007), Parker (2001). While this literature has many successes, for the aggregate stock market and value/growth stocks, it is still useful to study an arguably simpler, tractable model.

There could be two ways to "test", or explore, the model. The most direct and literal way would be to look at the behavior of assets during disasters. Barro (2006) led the way with the this analysis of stocks and bills during historical disasters. This paper is one more motivation to study how corporate bonds, and value and growth stocks have fared respectively during historical disasters. This is a potentially vast enterprise.

The second way to make the model testable is to work out predictions that should hold in a time series sample (such as the rich OECD countries since World War II) that have not experienced disasters. We obtain a series of time-series and cross-sectional predictions about bonds and stocks, that prima facie appears consistent with the main known facts. Hence, throughout the paper I try to highlight predictions that have been tested or could be tested in "normal times" samples.

Throughout the paper, I use the recently developed class of "linearity-generating" processes (Gabaix 2007). That class keeps all expressions in closed form. The entire paper could be rewritten with other processes (e.g. affine-yield models) albeit with considerably more complicated algebra, and the need to resort to numerical solutions. I suspect that the economics would be similar (the linearity-generating class and the affine class give the same expression to a first order approximation). Hence, there is little of economic consequence in the use of linearity-generating processes, and they should be viewed as simply an analytical convenience, that allows to explore many issues in a tractable way.

Section 2 presents the macroeconomic environment, and the cash-flow process for stocks and bonds. Section 3 derives the equilibrium prices. I next study in turn the model's implication for the predictability of returns, for stocks in section 5, and bonds in section 6. These results are useful for the calibration of the model, done in section 4.

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[3]In another interpretation of the model, the "disasters" are not macroeconomic disaster, but financial crises.

2 Model setup

2.1 Macroeconomic environment

The environment is a streamlined version of the one used by Rietz (1988) and Barro (2006). I consider an endowment economy, with C_t as the consumption endowment, and a representative agent with utility $V = \sum_t e^{-\delta t} C_t^{1-\gamma} / (1 - \gamma)$. Hence, the pricing kernel is $M_t = \partial V / \partial C_t = e^{-\delta t} C_t^{-\gamma}$, and the price of an asset yielding a dividends of D_t at time t is: $P_t = E_t \left[\sum_{s \geq t} M_s D_s \right] / M_t$, as per Lucas (1978) and Breeden (1979).

Following Rietz (1988) and Barro (2006), I state that each period $t+1$ a disaster may happen, with a probability p. If a disaster does not happen, $C_{t+1}/C_t = e^g$, where g is the normal-times growth rate of the economy. If a disaster happens, then $C_{t+1}/C_t = e^g B$, with $B > 0$.[4] For instance, if $B = 0.7$, consumption falls by 30%. To sum up:

$$\frac{C_{t+1}}{C_t} = \begin{cases} e^g & \text{if there is no disaster at } t+1 \\ e^g B_{t+1} & \text{if there is a disaster at } t+1 \end{cases} \tag{1}$$

As the pricing kernel is $M_t = e^{-\delta t} C_t^{-\gamma}$,

$$\frac{M_{t+1}}{M_t} = \begin{cases} e^{-R} & \text{if there is no disaster at } t+1 \\ e^{-R} B_{t+1}^{-\gamma} & \text{if there is a disaster at } t+1 \end{cases} \tag{2}$$

where

$$R = \delta + \gamma g_c$$

is the risk-free rate in an economy that would have a zero probability of disasters.

This complete description of the macroeconomic environment, and any asset can be priced . The innovation in this paper is to propose a way to model the time-varying riskiness of stocks, bonds and exchange rates.

2.2 Setup for Stocks

A given stock (there can be many stocks in this economy) has a dividend D_t, which follows:

$$\frac{D_{t+1}}{D_t} = \begin{cases} e^g \left(1 + \varepsilon_{t+1}^D\right) & \text{if there is no disaster at } t+1 \\ e^g \left(1 + \varepsilon_{t+1}^D\right) F_t & \text{if there is a disaster at } t+1 \end{cases} \tag{3}$$

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[4]Typically, extra i.i.d. noise is added, but given that it never materially affects the asset prices, it is omitted here.

where $\varepsilon_{t+1}^D > -1$ is a zero mean fluctuation that does not matter, except in the calibration of dividend volatility, that is independent of whether there is a disaster. In normal times, D_t grows at a rate g. But, if there is a disaster, the dividend of the asset is partially wiped out (as in Barro 2006): the dividend is multiplied by $F_t \geq 0$. F_t is the recovery rate of the stock. When $F_t = 0$, the asset is expropriated. When $F_t = 1$, there is no loss in dividend. To model the time-variation in the asset's recovery rate, I define:

$$H_t = p_t E_t \left[B_{t+1}^{-\gamma} F_t \right] - p_t \tag{4}$$

H_t can be called the "expected resilience" of the asset. When the asset is expected to do well in a disaster (high F_t), H_t is high – investors are optimistic about the asset.[5]

To streamline the model, I specify the dynamics of H_t directly, rather than by looking at the individual components, p_t, B_{t+1}, F_{t+1}. I split it H_t into a constant part H_* and a variable part $\widehat{H}_t$:

$$H_t = H_* + \widehat{H}_t$$

and postulate the following process for the variable part $\widehat{H}_t$:

$$\text{Linearity-Generating twist: } \widehat{H}_{t+1} = \frac{1 + H_*}{1 + H_t} \rho_H \widehat{H}_t + \varepsilon_{t+1}^H \tag{5}$$

where $E_t \varepsilon_{t+1}^H = 0$, and $\varepsilon_{t+1}^H, \varepsilon_{t+1}^D$, and whether there is a disaster, are uncorrelated variables[6] Eq. 5 means that $\widehat{H}_t$ mean-reverts to 0, but as a "twisted" autoregressive process. As H_t hovers around H_*, $\frac{1+H_*}{1+H_t}$ is close to 1, so that the process behaves much like a regular AR(1): $E_t \widehat{H}_{t+1} \sim \rho_H \widehat{H}_t$. The $\frac{1+H_*}{1+H_t}$ term is a "twist" term that makes the process very tractable. It is best thought as economically innocuous, and simply an analytical convenience.[7]

The above finishes the setup for stocks. I next turn to the bonds.

2.3   Setup for Bonds

I start with a motivation for the model. The most salient puzzles on nominal bonds are arguably the following. First, the nominal yield curve slopes up on average; i.e., long term rates are higher than short term rates. Second, there are bond risk premia. The risk premium on long term bonds increases with the difference in the long term rate minus short term rate. (Campbell Shiller 1991, Cochrane and Piazzesi 2005, Fama 2006, Fama and Bliss 1987).

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[5] This interpretation is not so simple in general, as H also increases with the probability of disaster.

[6] ε_{t+1}^H can be heteroskedastic – but, its variance need not be spelled out, as it does not enter into the prices. However, the process needs to verify $\widehat{H}_t \geq (\rho - 1)(1 + H_*)$, so the process is stable, and also $\widehat{H}_t \geq -p - H_*$ to ensure $F_t \geq 0$. Hence, that the variance needs to vanish in a right neighborhood $\max((\rho - 1)(1 + H_*), -p - H_*)$. Gabaix (2007) provides more details on the stability of Linearity-Generating processes.

[7] Gabaix (2007) provides a more thorough analyze of the linearity-generating twist.

I propose the following explanation. When a disaster occurs, inflation increases (on average). As very short term bills are essentially immune to inflation risk, while long term bonds lose value when inflation is higher, long term bonds are riskier, hence they get a higher risk premium. Hence, the yield curve slope up.[8]

Moreover, the magnitude of the surge in inflation is time-varying, which generates a time-varying bond premium. If that bond premium is mean-reverting, that generates the Fama-Bliss puzzle.

Note that this explanation is quite generic, in the sense that it does not hinge on the specifics of the disaster mechanism. The advantage of the disaster framework is that it allows for formalizing and quantifying the idea in a simple way.

Several authors have models where inflation is higher in bad times, which makes the yield curve slope up (Brandt and Wang 2003, Piazzesi and Schneider 2006, Wachter 2006). The paper is part of burgeoning literature on the economic underpinning of the yield curves, see e.g. Piazzesi and Schneider (forth.), Vayanos and Vila (2006), Xiong and Yan (2006). An earlier unification of many puzzles is provided by Campbell-Cochrane (1999) and Wachter (2006), who studies a Campbell-Cochrane (1999) model, and conclude that it explains an upward sloping yield curve and the Campbell-Shiller (1991) findings. The Brandt and Wang (2003) study is also a Campbell-Cochrane (1999) model, but in which risk-aversion depends directly on inflation.

I now formalize the above ideas. Inflation is i_t. The real value of the potential coupon is called D_t, and evolves as:

$$\frac{D_{t+1}}{D_t} = \begin{cases} 1 - i_t & \text{in normal times} \\ (1 - i_t) F & \text{if crisis} \end{cases} \tag{6}$$

In normal times, it depreciates at the rate of inflation, i_t. In disasters, there is possibility of default. A recovery rate $F = 1$ means full recovery, $F < 1$ partial recovery. The default could be an outright default (e.g., for a corporate bond), or perhaps a burst of inflation that increases the price level hence reduces the real value of the coupon (as in Barro 2006). In this first pass, to isolate the bond effects, I assume the case where $p(FB^{-\gamma} - 1) = H$ is a constant.

I decompose inflation as $i_t = i_* + \widehat{i}_t$, where i_* is its constant part, and $\widehat{i}_t$ is its variable part. The variable part of inflation follows the process:

$$\widehat{i}_{t+1} = \frac{1 - i_*}{1 - i_t} \cdot \left(\rho_i \widehat{i}_t + 1_{\{\text{Crisis at } t+1\}} \left(j_* + \widehat{j}_t \right) \right) + \varepsilon_{t+1}^i \tag{7}$$

This equation means first that, if there is no disaster, $E_t \widehat{i}_{t+1} = \frac{1-i_*}{1-i_t} \cdot \rho_i i_t$, i.e. inflation follows the Linearity-Generating (Appendix A) twisted autoregressive process. Inflation mean-reverts at a

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[8]Several authors have models where inflation is higher in bad times, which makes the yield curve slope up. See Brandt and Wang (2003), Piazzesi and Schneider (2006), Wachter (2006).

rate ρ_i, with the LG-twist $\frac{1-i_*}{1-i_t}$ to ensure tractability

In addition, in case of a disaster, inflation jumps by an amount $j_t = j_* + \widehat{j}_t$. This jump in inflation makes long term bonds particularly risky. j_* is the baseline jump in inflation, $\widehat{j}_t$ is the mean-reverting deviation from baseline. It follows a twisted auto-regressive process, and, for simplicity, does not jump during crises:

$$\widehat{j}_{t+1} = \frac{1 - i_*}{1 - i_t} \cdot \rho_j \widehat{j}_t + \varepsilon^j_{t+1} \tag{8}$$

This ends the description of the inflation process.

3 Equilibrium Asset Prices

The previous section described the process for three cash-flows processes for stocks, bonds and exchange rate. This section calculates their prices.

3.1 Stocks

I derive the price of a stock. Following the general procedure for Linearity-Generating processes (Appendix A), I start by forming, using (2) and (3):

$$\frac{M_{t+1}D_{t+1}}{M_t D_t} = \begin{cases} e^{-R+g}\left(1 + \varepsilon^D_{t+1}\right) & \text{if there is no disaster at } t+1 \\ e^{-R+g}B^{-\gamma}F_t\left(1 + \varepsilon^D_{t+1}\right) & \text{if there is a disaster at } t+1 \end{cases}$$

As the probability of disaster at $t+1$ is p, and using the definition of $H_t = p\left(B^{-\gamma}F_t - 1\right)$,

$$E_t\left[\frac{M_{t+1}D_{t+1}}{M_t D_t}\right] = e^{-R+g}\left((1-p)\cdot 1 + p\cdot B^{-\gamma}F_t\right) = e^{-R+g}\left(1 + H_t\right) = e^{-R+g}\left(1 + H_* + \widehat{H}_t\right) \tag{9}$$

Next, as $\widehat{H}_{t+1}$ is independent of whether there is a disaster, and is uncorrelated with ε^D_{t+1},

$$\begin{aligned} E_t\left[\frac{M_{t+1}D_{t+1}\widehat{H}_{t+1}}{M_t D_t}\right] &= E_t\left[\frac{M_{t+1}D_{t+1}}{M_t D_t}\right] E_t\left[\widehat{H}_{t+1}\right] = e^{-R+g}\left(1 + H_t\right)\cdot\frac{1 + H_*}{1 + H_t}\rho\widehat{H}_t \\ &= e^{-R+g}\left(1 + H_*\right)\rho\widehat{H}_t \end{aligned} \tag{10}$$

In (5), the reason for the $1 + H_t$ term in the denominator was to ensure that the above express would remain linear in $\widehat{H}_t$.

Eq. 9 and 10 ensure that $Y_t = M_t D_t\left(1, \widehat{H}_t\right)$ is a Linearity-Generating process (Appendix A), $E_t\left[Y_{t+1}\right] = \Omega Y_t$, with

$$\Omega = e^{-R+g}\begin{pmatrix} 1 + H_* & 1 \\ 0 & \rho\left(1 + H_*\right) \end{pmatrix}.$$

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The results in Appendix A give the stock price.

Proposition 1 *(Stock prices) The price of a stock i is:*

$$P_t = \frac{D_t}{1 - e^{-R+g}(1+H_*)} \left(1 + \frac{\widehat{H}_t}{1 - e^{-R+g}(1+H_*)\rho} \right) \tag{11}$$

In the limits of short time periods, writing $\rho = e^{-\phi_H}$ with R, g, ϕ_H small, the price is:

$$P_t = \frac{D_t}{r_i} \left(1 + \frac{\widehat{H}_t}{r_i + \phi_H} \right) \tag{12}$$

with

$$r_i = R - g - H_*. \tag{13}$$

When $\widehat{H}_t \equiv 0$, those are the expressions of Barro (2006). Hence Proposition 1 is the extension of Barro's result with a stochastic recovery rate, $\widehat{H}_t$. As always with a Linearity-Generating process, the details of the noise does not matter. Only a few moments matter for the price. For instance, one can specify many stochastic structures for the variance of $\widehat{H}_t$.

3.2 Bonds

To obtain bond prices, two notations are useful. First, define the risk premium:

$$\pi_t \equiv \frac{pB^{-\gamma}F}{1+H} \widehat{j}_t \tag{14}$$

π_t is the mean-reverting part of the "risk adjusted" expected increase in inflation if there is a disaster. It will be the variable part of the bond risk premium, hence its name. It is analogous to the $\widehat{H}_t$ term for stocks. Second, I parametrize the typical jump in inflation j_* in terms of a number $\kappa \leq (1 - \rho_i)/2$ (I assume not too large inflation jump j_*):[9]

$$\frac{pB^{-\gamma}Fj_*}{1+H} = (1 - i_*)^2 \kappa (1 - \rho_i - \kappa) \tag{15}$$

The next Proposition gives bond prices. Its proof is in Appendix C. In the continuous time limit, I use $\rho_i = e^{-\phi_i}$ and $\rho_\pi = e^{-\phi_\pi}$.

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[9] Normally, calculating bond prices in a Linearity-Generating system involves calculating the exponential of a matrix, hence finding its eigenvalues. To avoid having quadratic roots in the solution, I "presolve" the relevant equation, by parameterizing j_* by (15).

Proposition 2 *(Price of bonds) In the limits of small time intervals, with $H = pB^{-\gamma}F - p$, the nominal short term rate is*

$$r_t = R - H + i_t$$

and the price of a nominal zero-coupon bond of maturity T is given by:

$$Z_t(T) = e^{-(R-H+i_{**})T}\left(1 - \frac{1 - e^{-\phi_i T}}{\phi_i}(i_t - i_{**}) - \frac{\frac{1-e^{-\phi_i T}}{\phi_i} - \frac{1-e^{-\psi_\pi T}}{\psi_i}}{\psi_\pi - \phi_i}\pi_t\right). \tag{16}$$

with $i_{**} \equiv i_* + \kappa$ and $\psi_\pi \equiv \phi_\pi - \kappa$. . The expression for discrete time case is given in Eq. 54 of Appendix C.

Proposition 2 is an important result of this paper. It will allow a closed-form expression for the yield curve, derived from an economic model.

To interpret it, it is good to have closed forms for two key variables about bonds, forward rates and yields.

Proposition 3 *(Bond yields and forward rates). The bond yield is, by definition, $y_t(T) = -(\ln Z_t(T))/T$, with $Z_t(T)$ given by (16). The forward rate, $f_t(T) \equiv -\partial \ln Z_t(T)/\partial T$ is:*

$$f_t(T) = R - H + i_{**} + \frac{e^{-\phi_i T}(i_t - i_{**}) + \frac{e^{-\phi_i T} - e^{-\psi_\pi T}}{\psi_\pi - \phi_i}\pi_t}{1 - \frac{1-e^{-\phi_i T}}{\phi_i}(i_t - i_{**}) - \frac{\frac{1-e^{-\phi_i T}}{\phi_i} - \frac{1-e^{-\psi_\pi T}}{\psi_i}}{\psi_\pi - \phi_i}\pi_t}$$

They admit the Taylor expansions:

$$\begin{aligned}
f_t(T) &= R - H + i_{**} + e^{-\phi_i T}(i_t - i_{**}) + \frac{e^{-\phi_i T} - e^{-\psi_\pi T}}{\psi_\pi - \phi_i}\pi_t + O(\varepsilon^2) & (17) \\
&= R - H + i_{**} + \left(1 - \phi_i T + \frac{\phi_i^2 T^2}{2}\right)(i_t - i_{**}) + \left(T - \frac{\phi_i + \psi_\pi}{2}T^2\right)\pi_t + h.o.t. & (18)
\end{aligned}$$

and

$$\begin{aligned}
y_t(T) &= R - H + i_{**} + \frac{1 - e^{-\phi_i T}}{\phi_i T}(i_t - i_{**}) + \frac{\frac{1-e^{-\phi_i T}}{\phi_i} - \frac{1-e^{-\psi_\pi T}}{\psi_i}}{(\psi_\pi - \phi_i)T}\pi_t + h.o.t. & (19) \\
&= R - H + i_{**} + \left(1 - \frac{\phi_i T}{2} + \frac{\phi_i^2 T^2}{6}\right)(i_t - i_{**}) + \left(\frac{T}{2} - \frac{\phi_i + \psi_\pi}{6}T^2\right)\pi_t + h.o.t. & (20)
\end{aligned}$$

3.3 Expected returns of assets

I state a general Proposition about the expected returns in this economy (which the reader may skip in a first reading).

Proposition 4 *(Expected returns) Consider an asset, and call $P_t^{\#} = E_t[P_{t+1} + D_{t+1} \mid \text{Disaster at } t+1]$, the value that the asset would have if a disaster happened at time $t+1$. Then, the expected return of the asset at t, conditional on no disasters, is:*

$$r_{e,t} = \frac{1}{1-p}\left(e^R - pB^{-\gamma}\frac{P_t^{\#}}{P_t}\right) - 1 \tag{21}$$

In the limit of small time intervals (continuous time),

$$r_{e,t} = R + p\left(1 - B^{-\gamma}\frac{P_t^{\#}}{P_t}\right) \tag{22}$$

$$= R - p\left(B^{-\gamma} - 1\right) + pB^{-\gamma}\left(1 - \frac{P_t^{\#}}{P_t}\right) \tag{23}$$

where $R - p\left(B_t^{-\gamma} - 1\right)$ is the risk-free rate in the economy.

Proof. The Euler equation, $1 = E_t[R_{t+1} \cdot M_{t+1}/M_t]$, gives:

$$1 = e^{-R}\left[(1 - p_t)(1 + r_{e,t}) + p_t\left(B^{-\gamma}\frac{P_t^{\#}}{P_t}\right)\right]$$

hence (21). The continuous time expression comes from taking the limit of (21) to 0 of R, p_t, r_{et}. ■

The unconditional expected return on the asset, on an infinite sample that includes disaster, is (in the continuous time limit)

$$r_e + p_t\left(\frac{P_t^{\#}}{P_t} - 1\right) = R - p_t\left(B_t^{-\gamma}F_t - 1\right) - p_t E_t\left(B_t^{-\gamma} - 1\right)\left(\frac{P_t^{\#}}{P_t} - 1\right)$$

When $B_t^{-\gamma}$ is large, $B_t^{-\gamma} - 1$ and $B_t^{-\gamma}$ are close. So, as observed by Barro (2006), the unconditional expected return, and the expected return conditional on no disasters are very close. The possibility of disaster affects mostly the risk premium, and much less the expected loss.

Formula (21) indicates that only the behavior in disasters (the $P_{t+1}^{\#}/P_t$ term) creates a risk premium. It is equal to the risk-adjusted (by $B_t^{-\gamma}$ – the probability is augmented by the relative importance of the event in terms of marginal utility of consumption) expected capital loss of the asset if there is a disaster. That makes the analytics simple, as only $E_t\left[P_t^{\#}/P_t - 1\right]$ is needed to obtain a cross-section of risk premia.

4 A Calibration

This section is more applied that the rest of the paper, so the reader may want to skip it in the first reading. Units are yearly.

4.1 Parameter values

Preferences. For the time-preference, $\delta = 4\%$, and for risk aversion $\gamma = 4$.

Macroeconomy. In normal times, consumption grows at rate $g_c = 2.5\%$. The probability of disaster is $p = 1.7\%$, as estimated by Barro (2006). In disasters, the recovery rate of consumption is $B = 0.6$. [10]

Stocks. The volatility of the dividend is $\sigma_D = 11\%$, as in Campbell and Cochrane (1999).

To specify the volatility of the recovery rate F_t, I specify that it has a baseline value $F_* = B$, and support $F_t \in [F_{\min}, F_{\max}] = [0, 1]$. That is, if there is a disaster, stocks can do anything between losing all their value and losing no value. The speed of mean-reversion $\phi = 0.15$, which gives a high-life of 4.6 years, and is in line with various estimates from the predictability literature. Given these ingredients $(F_{\min}, F_{\max}, F_*, \text{and } \phi)$, Appendix D specifies volatility process for F_t, and calculates the average unconditional volatility of F_t. I use this procedure for the recovery rate of stocks, but also bonds and exchange rates.

Bonds. Inflation is persistent, so I take $\phi_i = 8\%$. I keep $\phi_\pi = 15\%$ for the speed of mean-reversion of the magnitude of inflation risk. I calibrate $j_* = 2\%/\left(5pB^{-\gamma}\right) = 3\%$, and $j_t \in [-3\%, 9\%]$.[11]

4.2 Implications for levels and volatilities

I now turn to the average value of various economic quantities of interest. I differ the conclusion on the predictability of the assets to the next sections.

T-bills. The short-term rate is 0.9% ($r_{ST} = R - p\left(B^{-\gamma}F_\$ - 1\right)$), assuming no inflation burst in a disaster (that is, $F_\$ = 1$ for government T-bills).

Stocks. The normal-times expected returns on equities is $R_e = R - p\left(B^{-\gamma}F_* - 1\right) = 7.8\%$, which corresponds to an equity premium of 6.3%. The unconditional expected return on equity (i.e., in long samples that include disasters) is $R_e - pF_* = 5.6\%$. The difference between those two returns is 1%, so as in Barro (2006), most of the return on equity comes from a risk premium.

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[10] In Barro, the recovery rate is a stochastic $\widetilde{B}$. My number matches it in the sense that $B^{1-\gamma} = E\left[\widetilde{B}^{1-\gamma}\right]$. Note that the average disaster could have a much higher recovery rate than 0.6, as the B reflects the possibility of some very bad disasters.

[11] I target a range for the 10 year spread in the yield curve $(y\,(10) - y\,(0))$ equal to $[-2\%, +6\%]$, for a baseline value of 2%. By Eq. 20, $y\,(10) - y\,(0) \simeq 5\phi_i\,(\kappa + \pi_t) = 5pB^{-\gamma}F\,(j_* + j_t)$.

The price/dividend ratio is $P/D = 18.7$ (that's eq. 12, evaluated at $\widehat{H}_t = 0$) in-line with the empirical evidence (see e.g. the number in Campbell Cochrane, 1999, Table 1).

The standard deviation of $\ln(P/D)$ is: 0.22. Campbell and Cochrane (1999) report an empirical standard deviation of 0.27.

The volatility of the equity premium is $\sigma_H = 1.0\%$ per year. This translate into a volatility of the log of the price / dividend ratio equal to 6.5%.[12] The total equity volatility depends on the correlation between dividend growth and the asset resilience, H_t.[13] With a correlation of 0, 0.5 and 1, the resulting equity volatility is, respectively: 12.7%, 15.3%, and 17.5%. This is in line with the empirical U.S. estimate, about 15% per year.[14]

I conclude that the model can quantitatively account for an "excess" volatility of stocks. In this model, this is due to the stochastic risk-adjusted intensity of disaster.

Bonds. The typical slope of the 10 year rate, $y(10) - y(0)$, is 2%. The annual volatility of that slope is 0.6% (this is, $\xi(\text{Max spread} - \text{Min spread}) \cdot \phi_\pi^{1/2}$).

I next turn to the predictability generated by the model.

5 Return predictability in stocks

Applying Proposition 4 gives the expected stock returns.

Proposition 5 *(Expected returns on stocks) The expected returns on stock i, conditional on no disasters, are:*

$$R_t^e = R - H_* - \widehat{H}_t \tag{24}$$

Proof. If a disaster happens, dividends are multiplied by $B_t F_t$ (which is less than 1 in a disaster). As $\widehat{H}_t$ does not change, $P_t^\#/P_t = F_t$. So, returns are, by Eq. 22,

$$R_t^e = R + p_t\left(1 - B_t^{-\gamma} F_t\right) = R - H_t = R - H_* - \widehat{H}_t.$$

∎

5.1 Partial predictability of aggregate stock market returns

Consider (12) and (24). We think about the aggregate stock market, for which H_* is a fixed quantity. When $\widehat{H}_t$ is high, (24) implies that the risk premium is low, and P/D ratios (12) are high. Hence, the

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[12] Also, in a sample with rare disasters, changes in the P/D ratio mean only changes in future returns, not changes in future dividends. This is in line with the empirical findings of Campbell and Cochrane (1999, Table 6).

[13] It is easy to impose such a correlation in the model.

[14] If their is a positive correlation between innovation to D and innovations to F, the volatility can be higher.

model demonstrates that when the market-wide P/D ratio is low, stock market returns will be higher than usual. This is the view held by a number of reputable financial economists (e.g. Campbell and Schiller 1988, Cochrane 2006, Boudoukh, Richardson and Whitelaw 2006), although the view is still controversial (Goyal and Welch 2006).

The model predicts the following magnitudes for regression coefficients. I call $r_{t \to t+T}$ the return from holding the asset from t to $t + T$

$$E_t [r_{t \to t+T}] = \alpha + \beta \ln (D/P)_t$$

then (for small to moderate Δt's, where Taylor approximations hold) the slope is: $\beta = (R_i + \phi) T$ i.e. about $\beta \simeq 0.2$ with annual predictability This is in line with the value estimate of Lettau and Van Nieuwerburgh (forthcoming).

In the model, the consumption / aggregate wealth ratio is: $CAY_t = (R - H_*) / \left(1 + \frac{\widehat{H}_t}{R+\phi}\right)$. Hence, CAY predicts future returns, as in Lettau and Ludvigson (2001). In the regression: $E_t [r_{t \to t+T}] = \alpha + \beta \ln CAY_t$, the coefficient would be the same $\beta = (R_i + \phi) T$.

5.2 Stocks: Value and growth stocks

If a disaster happens, different stocks will fare differently.[15] Their dividend will change by F_t, where F is the recovery rate. This dispersion of sensitivity of dividends to disasters leads to a dispersion of premia and prices in normal times. I propose that this is a fruitful way to think about value and growth stocks (Fama French 1996, Lakonishok, Shleifer, Vishny 1994). This is a small variant on the idea that the value premium might be a compensation for "distress risk" (Fama French, Campbell et al. 2006). Here distress happens during the rare, economy-wide disasters.

First, consider the simple case of constant $F_t = F_*$. Stocks with a low F_* are "risky ", as they will perform poorly during disasters. They also have a low H_* (Eq. 4), and by Proposition 1, they have a low price/dividend ratio. They look like "value" stocks. By Proposition 5, they have high returns – a compensation for their riskiness during disasters.

The same reasoning holds if the F_t is variable.[16] Stocks with a low $\widehat{F}_t$ are risky, have low $\widehat{H}_t$, low P/D ratio, and high future returns. They are value stocks.

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[15] For instance, stocks with a lot of physical assets that might be destroyed, or stocks very reliant on external finance, might have a lower F.

[16] Fama French (2006) show that the value premium is essentially due (at a mechanical level) to "migration", i.e. mean-reversion in P/E ratio: a stock with high (resp. low) P/E ratio tends to see it's P/E ratio go up (resp. down). It terms of the model, this means that H_* is relatively constant across stocks, while the fluctuations in $\widehat{H}_t$ drive the value premium.

Some thought experiments This perspective suggests a few thought experiments. Suppose that the expected recovery rates, F_t, are not variable, i.e. $H_t = H_*$. Covariances between stocks happen only because of covariances between cash flows D_t in normal times. Hence, the stock market betas will only reflect the "normal times" covariance in cash flows. But risk premia are only due to the behavior in disasters, H_*. Hence, there will be no causal link between betas, stock market beta, and returns. The "normal times" betas could have no relation with risk premia. However, there could be some spurious links if, for instance, stocks with low H_* had higher cash-flow betas. One could conclude that cash-flow beta commands a risk premium, but this is not because cash-flow beta causes a risk premium. It is only because stocks with high cash-flow beta happen to also be stocks that have a large loading on the disaster risk.[17]

This experiment may help explain the somewhat contradictory findings in the debate of whether characteristics or covariances explain returns (Daniel and Titman 1997, Davis Fama French 2000).

With an auxiliary assumption, the model can also explain the appearance of a "value factor", such as the High Minus Low (HML) factor of Fama and French. Suppose that:

$$\widehat{H}_{it} = \beta_i^H \widehat{H}_{Mt} + \widehat{h}_{it}$$

where $\widehat{H}_{Mt}$ is a systematic ($m$arket-wide) part of the expected resilience of the asset, and $\widehat{h}_{it}$ is the idiosyncratic

Consider two benchmarks. If for all stocks $\beta_i^H = 1$ (the "characteristics benchmark") so that all dispersion in $\widehat{H}_i$ is idiosyncratic, then characteristics (the P/D ratio of a stock) predict future returns, but covariances do not. On the other hand, if for all stocks $\widehat{h}_{it} \equiv 0$, but the β_i^H vary across stocks, all expected returns are captured by a covariance model. In general, reality will be in between, and covariances and characteristics are both useful to predict future returns. This is generally what empirical studies find (e.g. Jagannathan and Wang forth.). Also, we see how for some samples or time-periods, characteristics may work better than covariances (Daniel and Titman 1997) or vice-versa (Davis, Fama, French 2000).

In the model, the value spread forecasts the equity premium and the value premium Consider a period of "exuberance", high $\widehat{H}_t$, where the dispersion of P/D ratios is high. What about future returns? Future returns of the market will be low. Also, value stocks are going to do relatively better than growth stocks, so that that Fama-French factor HML (High minus Low, the return of stock with a high book/market ratio, minus stocks with a low book/market ratio) will be high.

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[17]Various authors (Julliard and Parker 2005, Campbell and Vuolteenaho 2006, Hansen Heaton and Li 2006) find that value stocks have higher long run cash flow betas. It is at least plausible that stocks that have high cash-flow betas in normal times also have high cash-flow betas in disasters, i.e. a low F_t and a low H_t. But is is their disaster-time covariance that creates a risk premium, not the normal-time covariance.

Hence, the model predicts that when the dispersion (standard deviation, or interquartile range) of the P/D, or of the Market/Book of stocks, is high, then HML_{t+1} should be high, and $(R_M - R_f)_{t+1}$ should be low. The predictions are in qualitative agreement with the findings of Liu and Zhang (2006). When "optimism" $\widehat{H}_t$ is high, then the Market to Book spread (the value of M/B for the top decile of stocks sorted by M/B, minus the bottom decile of stocks with the same sorting), and the Book to Market spread is low. Additionally, future aggregate returns are low.

6 Bond Premia and Yield Curve Puzzles Explained by the Model

The following proposition states the expected bond returns.

Proposition 6 *(Expected returns on bonds) Conditional on no disaster, the real return on the bond of maturity T is:*

$$R_e(T) = R - H_* + \frac{\frac{1-e^{-\phi_i T}}{\phi_i}\left(\kappa\left(\phi_i - \kappa\right) + \pi_t\right)}{1 - \frac{1-e^{-\phi_i T}}{\phi_i}\left(i_t - i_{**}\right) + \frac{\frac{1-e^{-\phi_i T}}{\phi_i} - \frac{1-e^{-\psi_\pi T}}{\psi_i}}{\psi_\pi - \phi_i}\pi_t} \tag{25}$$

In the case of small deviations, and $\phi_i T \ll 1$,

$$R_e(T) = R - H_* + T\left(\kappa\left(\phi_i - \kappa\right) + \pi_t\right) + O\left(\varepsilon^2\right) \tag{26}$$

so that the excess return on the T maturity bond is:

$$R_e(T) - R_e(0) = T\left(\kappa\left(\phi_i - \kappa\right) + \pi_t\right) + O\left(\varepsilon^2\right) \tag{27}$$

Proof. After a disaster, π_t does not change, but i_t jumps to $i_t + b + i_t^{\#}$. That creates a capital loss equal to:

$$V_t - V_t^{\#} = e^{-(R-H+i_{**})T} \cdot \frac{1 - e^{-\phi_i T}}{\phi_i}\left(b + i_t^{\#}\right)$$

for the bond holder. Lemma 4 gives the risk premia, using $pB^{-\gamma}\left(b + i_t^{\#}\right) = \kappa\left(\phi_i - \kappa\right) + \pi_t$. The approximate result comes from the fact that $\frac{1-e^{-\phi_i T}}{\phi_i} \sim T$ when $T\phi_i \to 0$. ∎

We can now extract economic meaning from Proposition 2.

6.1 Typical behavior of the yield curve

First, as is natural the bond price decreases in i_t, and in π_t.

Bond carry a risk premium   Consider (27). It indicates that, the longer the bond maturity, the higher the risk premium. Furthermore, the risk premium is approximately proportional to the maturity T of the bond. This is the finding of Cochrane and Piazzesi (2005).

The nominal yield curve slopes up on average   Let us analyze the long term rate (74), first when there is no stochasticity, and $i_t = i_*$, and $i_t^{\#} = 0$. The short term rate is $r = R - H + i_*$, while the long term rate (formally defined as $R = -\lim_{T \to \infty} \ln Z_t(T)/T$), is $R = r + \kappa > r$. Hence, the long term rate is above the short term rate, by an amount $\kappa > 0$.[18] When the disaster happens, inflation jumps by $b = \kappa(\phi_i - \kappa)/(pB^{-\gamma}) > 0$, which does not affect the value of the very short term bill, but does depress the value of long maturity bonds. Long maturity bonds are riskier, so they command a risk premium. Hence, the long term rate is higher than the short term rate.

To see the magnitude of the effect; say that the average slope of the yield curve is $\kappa = 3\%$. With $\phi_i = 10\%$, $R = 10\%$, and $pB^{-\gamma} = 8\%$, this gives an average increase in inflation during disasters of: $b = \kappa(\phi_i - \kappa)/(pB^{-\gamma}) = 3\%$. This is not implausible, given that b is a number which we know very little about. In any case, the model easily generates a high slope of the yield curve.

6.2   Predictability of bond excess returns

In this section I show how the model matches the key findings of the research on the predictability in the yield curve. In this, the model is similar to the econometric frameworks of Duffee (2002) and Dai and Singleton (2002), who show how the Fama-Bliss and Campbell-Shiller results can be accounted by affine models. The main advantage is that the present model is a microfounded economic model of the term structure.

6.2.1   Predictability with the forward spread (Fama-Bliss)

Fama-Bliss (1987) regress excess returns on the forward rate minus the short-term rate:

$$\text{Fama-Bliss regression: Excess return on bond of maturity } T = a + \beta \cdot (f_t(T) - r_t) \qquad (28)$$

A model with constant risk premia (e.g., the expectation hypothesis) would predict $\beta = 0$. On the other hand, if the present model is right, the above regression should yield a slope $\beta = 1$. [19] This

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[18] If inflation fell during disasters, then we would have $\kappa < 0$, and the average nominal yield curve would slope down.

[19] In terms of the model, the excess return on $T-$maturity bond is approximately $T(\kappa(\phi_i - \kappa) + \pi_t)$ (see eq. 27), while the forward spread is $f_t(T) - f_t(0) \simeq T\pi_t$ (see Eq. 18). Hence, the regression (28) is: $T(\kappa(\phi_i - \kappa) + \pi_t) = a + \beta \cdot T\pi_t$, which yields $\beta = 1$.

value $\beta = 1$ is precisely what Fama and Bliss have found, a finding confirmed by later research (Cochrane and Piazzesi 2005, Fama 2006). This is quite heartening for the model.[20]

Economically, the finding, with a slope coefficient of $\beta = 1$, means that most of the variations in (yield minus spot rate) are due to variations in risk-premium.

I conclude that the model explains the findings of Fama-Bliss (1987).

6.2.2 Predictability with the slope of the yield curve (Campbell Shiller)

Campbell and Shiller (1991) regress changes in yields on the spread between the yield and the short-term rate:

$$\text{Campbell-Shiller regression I: } \frac{y_{t+\Delta t}(T - \Delta t) - y_t(T)}{\Delta t} = a + \beta \cdot \frac{y_t(T) - y_t(0)}{T} \qquad (29)$$

The expected hypothesis predicts $\beta = 1$. This paper's model predicts (see Appendix C), in the limit $\phi_i^2 var(i_t)/var(\pi_t) \ll 1$,[21] $\beta = -(1 + \phi_\pi T)$.

Campbell and Shiller find negative β's, with a roughly affine shape as a function of maturity, which is broadly a success for the model. The model predicts $\beta = -1$ at very short maturities, while Campbell-Shiller find rather $\beta \simeq 0$. This is probably cause by a small bit of predictability of the interest rate movements in the very short term, which pushes the β away from -1, and toward $+1$. If the model had the short-predictability of the spot interest rate, it would presumably match the Campbell-Shiller fact, but I think it is better not to add that feature to the model at this stage.

To understand the economics, I use a Taylor expansion. The slope of the yield curve is, to a 0-th order approximation in T:

$$\text{Slope} \equiv (y_t(T) - y_t(0))/T = \frac{1}{2}(-\phi_i i_t + \pi_t) + o(T)$$

The first term, $-\phi_i i_t$, reflect the "expectation hypothesis" term: it captures the predictable movements in the short term rate (here, its mean-reversion at rate ϕ_i). The second term reflects the bond risk premium. Hence, when regressing the future movements in the short term rate (equal to $-\phi_i i_t/2$) on the slope, one gets the "expectation hypothesis" sign. Indeed, there was no sign premium, we

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[20]Other models, if they have a time-varying bond risk premium proportional to the maturity of the bond, would have a similar success. So, the model illustrates a generic mechanism that explains the Fama-Bliss result.

[21]Eq. 20 gives, in the limit of $\phi_i \to 0$, and $\Delta t \to 0$:

$$\frac{y_{t+\Delta t}(T - \Delta t) - y_t(T)}{\Delta t} \simeq E_t[dy_t(T)]/dt - \partial y_t(T)/\partial T$$

$$= \left[-\frac{1}{2}T\phi_\pi \pi_t\right] - \left[+\frac{1}{2}\phi_\pi \pi_t\right] = -\frac{1 + T\phi_\pi}{2}\pi_t$$

while $(y_t(T) - r_t)/T \simeq \pi_t/2$. So their regression is, approximately: $-(1 + T\phi_\pi)\pi_t/2 = a + \beta \cdot \pi_t/2$, so that the regression coefficient should be approximately $\beta = -(1 + \phi_\pi T)$.

would have $\beta = 1$. That explains why the slope of the yield curve predicts movements in the future short term rate with the "correct" sign (from the point of view of the expectation hypothesis).

Futhermore, Campbell-Shiller find that the yield spread does have a predictive power for the short rate. The model predicts that too. The reason is that on the right-hand side of the Campbell-Shiller regression, one finds the slope of the yield curve:

$$\text{Campbell-Shiller regression II} \quad :$$
$$\text{Average change in the short rate over the next } T \text{ periods} \quad = \quad a + \beta^{II} \cdot \frac{y_t(T) - y_t(0)}{T}$$

Campbell and Shiller find a positive β^{II}, consistent in sign with the expectation hypothesis, which predicts $\beta^{II} = 1$. The model predicts:

$$\beta^{II} = \frac{\phi_i^2 var(i_t)}{\phi_i^2 var(i_t) + var(\pi_t)} \in (0, 1)$$

Indeed, the expected change in the short-term rate over T periods is $-\phi_i i_t T/2$. Hence, the model predicts that the slope of the yield curve partly predicts future movements of the short rate. The model correctly delivers the positive β that Campbell-Shiller found.

I conclude that the model can account for the main qualitative findings of Campbell Shiller (1991). For an even better fit, it would be better to add some short-term predictability in the model (as Dai and Singleton 2002), but for parsimony I leave that to future work.

6.2.3 Understanding the findings of Cochrane and Piazzesi (2005)

Cochrane and Piazzesi (2005, 2006) deepen the findings of Fama-Bliss and Campbell-Shiller. They establish that a parsimonious description of bond premia is given by: (i) 1 risk factor, such that (ii) a bond of maturity T has a loading proportional to T and (iii) this risk premium is well-captured by a "tent-shape" of forward rates, that capture the concavity but is roughly independent of the level or the slope of the forward curve.

The theory in this paper (assumes) (i), a single priced risk factor. Less trivially, the excess bond return is given by (27), i.e. a 1 factor model. Recall that $\kappa(\phi_i - \kappa)$ is a constant, while π_t is the time-varying part of the risk premium.

Do we obtain the tent shape? Recall that Cochrane and Piazzesi find that a good approximation for the risk premium is $\Pi_t = \sum_{T=0}^{4} \omega_T f_t(T)$, with ω_T weights that have a tent-shape, and Π_t is (to a good degree of approximation) independent of the level and the slope of the curve f_t. That means

that $\omega \cdot f$ depends on the curvature of $f(T)$, i.e. of $f''(T)$. Using Eq. 18, we find:

$$\text{Level of the forward curve } f(0) \quad : \quad R - H + i_t \tag{30}$$

$$\text{Slope of the forward curve } f'(0) \quad : \quad -\phi_i(i_t - i_{**}) + \pi_t \tag{31}$$

$$\text{Curvature of the forward curve } -f''(0) \quad : \quad -\phi_i^2(i_t - i_{**}) + (\phi_i + \psi_\pi)\pi_t \tag{32}$$

So, if an econometrician wanted to approximate π_t with the level, slope or curvature, what would be the best approximation? Again, we think of ϕ_i small. By (31), the slope of the forward curve is not a bad approximation. Unfortunately, it does contain a pollution term $\phi_i(i_t - i_{**})$. But (32), the curvature of the forward curve is a much better approximation, because the term in $i_t - i_{**}$ is multiplied by ϕ_i^2. This is why, in the view of the present theory, Cochrane and Piazzesi find that the curvature of the forward curve is the best way to approximate the risk premium.

We conclude that the simple model explains in a natural way the results of Fama Bliss, Campbell Shiller, and Cochrane and Piazzesi.

6.3   Further remarks on bonds

Government deficits, Ricardian equivalence, central bank independence, and the level of real interest rates   With additional, plausible assumptions, the model allows to think about further things. Consider the impact of the government Debt/GDP ratio, or of deficits (if current deficits predict a debt/GDP ratio later). It is plausible that if the Debt/GDP ratio is high, then, if there is a disaster, the government will sacrifice monetary rectitude (that could be microfounded), so that j_* is high, i.e. κ is higher. That implies that *when the Debt/GDP ratio (or the government deficit / GDP) is high, then long-term rates are higher, and the slope of the yield curve is steeper* (controlling for inflation, and expectations about future inflation in normal times). Dai and Philippon (2006) present evidence consistent with that prediction.

We note that this effect works in an economy where Ricardian equivalent holds. Higher deficits do not increase long term rates because they "crowd out" investment, but instead because they increase the temptation by the government to inflate away the debt if there is a disaster, hence the risk premium on government debt and real long term rates.

Likewise, say that an independent central bank is a more credible commitment not to increase inflation during disasters (π_t smaller, and b of κ smaller). Then, an independent central bank has a lower level of *real* long term interest rates. If an independent central bank means a lower F_t for bonds, it also means a lower level of real short term rates as well, though we can expect the effect to be smaller.

The Corporate Spread   Consider the corporate spread, which is the difference between the rate on corporate bonds and treasury bonds. For short term securities, there is no inflation risk, and the risk is entirely a default risk. The spread for short term securities:

$$\text{Corporate spread}_t^{\text{Short term}} = p_t B^{-\gamma} F_{\$t} \left(1 - F_{Corp,t}\right) \tag{33}$$

Hence, the corporate spread is equal to the expected default $p_t \left(1 - F_{Corp,t}\right)$, times a risk-adjustment term equal to $B^{-\gamma}$. Given $B^{-\gamma} \simeq 5$ in the calibration, the model proposes why corporate spread is so high, compared to historical U.S. values (e.g. Huang and Huang 2003). The corporate sector defaults during very bad states of the world, so that risk-adjusted probability of default is much higher than the physical probability of default.

We may also explain the finding of Krishnamurthy and Vissing-Jorgensen (2007), that when the debt/GDP ratio is high, the corporate spread is low; a finding for which their favored interpretation is a liquidity demand for treasuries. In the view of the present paper, one could say that, when Debt/GDP is high, the temptation to default via inflation (should a risk occur), is high, so $F_{\$t}$ is low, hence the corporate spread is low.

7   Options and tail risk

[This section is very preliminary] The price of a European put on a stock, with strike K is: $V_t = E\left[M_{t+T}\left(K - S_{t+T}\right)^+\right]/M_t$. In a time interval T, the probability of a crash $p_t T$, and if a crash happens at all, it will typically happen just once. So, for a deep out-of the money put, the value of the put (say K more than two standard deviations below the stock price, i.e., with σ the standard deviation of the stock price), with $S_t e^{-2\sigma T^{1/2}} > K > S_t V F_t$ is approximated by:

$$\widetilde{V}_t = p_t T \cdot B_t^{-\gamma} \left(K - S_t F_t\right) \tag{34}$$

The above eq. predicts that the put price is (i) linear in the time-to maturity T (ii) affine in the strike K.

It suggests the following procedure to estimate tail risk. One runs the equation, in the cross-section of T's or K's (one cross-section of K's is enough): $V_t\left(T, K\right) = aKT + bS_t T$. That identifies: $a = p_t B_t^{-\gamma}$ and $b = -p_t B_t^{-\gamma} F_t$, so we can have estimates of the intensity-adjusted probability of crashes $p_t B_t^{-\gamma}$, and expected intensity of disasters, $p_t B_t^{-\gamma} F_t$. As the expected return on the stock is $R - p_t B_t^{-\gamma} F_t$.

The above analysis shows that the stocks with a higher put price (control for "normal times" volatility) should have a higher risk premium, hence higher future expected returns. The model even

predicts a coefficient of 1 on the put price, after controlling (which is difficulty) for "normal times" volatility. Evaluating this prediction would be most interesting.

Empirically, Du (2006) provides an interesting analysis of the price of puts under disaster risk. He finds that a consumption jump size of 24% is required to explain deep out of the money puts. A jump of 48% creates a too high price for puts. He then observes that to explain the high price of at the money options, one needs some excess volatility, which he models by a habit formation of the type of Campbell Cochrane and Santos Veronesi. I suspect that with the excess volatility of the present paper (due to time-varying severity of disasters), the same calibration would hold.

The model predicts that a high price of deep out-of the money put predicts future high returns. It is already known that high price of at-the-money options (as proxied by the VIX index) predicts high future returns (Doran et al. 2006, Giot 2005, Guo and Whitelaw, 2006). So by continuity, this is likely that deep out of the money put prices will also predict high returns, though that specific prediction has yet to be tested.

8 Discussion

8.1 A behavioral interpretation of the model

While the model is presented as rational, it admits a behavioral interpretation. The varying beliefs about the probability and intensity of crashes could be rational, or behavioral, after all. Given that there are so few data points on crashes, there is little that constrains beliefs. This interpretation allows a revisit to several themes of the literature.

Sentiment. A high H_t increases stock prices. The model generates predictions analogous to the findings of the behavioral literature. For instance, Baker and Wurgler (2006, and forth.) find that periods of high (resp. low) sentiment are followed by low (resp. high) returns. This is exactly what the model generates. Also, they find that the effect is more pronounced in small firms. If small firms have a more volatile H_t, hence a higher "sentiment beta", this is also what we expect.

The model offers a coherent way to think about the joint behavior of sentiment and prices. This is not a trivial task. Otherwise, suppose we know a stochastic path of future sentiment, what should happen to the stock price? This is a priori a difficult problem that the model's structure allows to solve.

Overreaction Suppose that, when there is a positive innovation to dividends, investors also believe that the stock will do better if there is a crash, i.e. H_t is high. That makes the price/dividend ratio increase, and lower future returns. The price increases by more than the dividend, which is interpreted in the language of behavioral economics as "overreaction".

Stock market crashes   A shock to $\widehat{F}_t$ or $\widehat{p}_t$ is a discount rate shock, that is not related to consumption shock.   Hence, the model admits stock market crashes, that have no link with consumption news, such as the October 1987 crash. Such crashes of asset prices without changes in consumption are typically a problem for entirely consumption-based models, e.g. Bansal and Yaron (2004).

Twin stocks   "Twin stocks" give the same dividends, in two different countries. If markets are integrated, they should have the same price. However, they typically do not (Froot and Dabora 1999). This phenomenon can be accommodated in the model, if we think that, during disasters, the dividends will be different. For instance, in the BP/Shell example it could be that different tax rates or expropriation rates will affect the two stocks differently – the stocks have a different recovery rate F_t. This perspective further predicts that BP will covary more with the UK stock market, and Shell will covary more with the Dutch stock market – because changes in the catastrophe severity F_t in the UK affects both the whole UK market, and BP. On the other hand, the model predicts that fluctuations in the "expected growth rate of dividend" terms will not affect the twin stock spread. Hence, the difference in F_t across countries could be proxied by the twin stock spread.

Nominal illusion and asset prices   In the post-war U.S. data, times with high inflation are also times of low real stock prices. Modigliani and Cohn have proposed that this was due to nominal illusion, a view for which that Cambell, Vuolateenaho and coauthors have found empirical support (see also Brunnermeier and Julliard 2006 on housing prices). The present model proposes an alternative explanation. In times of high inflation, investors are "pessimistic" about stocks ($\widehat{H}_t$ is low), so indeed stock market valuations are low.

8.2   The whole closed economy: GDP, stocks, bonds

[This section is very preliminary, and should be skipped in a first reading]

To complete the painting of the economy proposed by the model, we want a way to model GDP, while still keeping the nice feature of the Lucas endowment economy. Appendix B presents one such model, that yields exactly the consumption process we have in our endowment economy, but with a cyclical GDP: $Y_t = C_t (1 + a + u_t)$, where a is some positive number, and u_t is a process with mean 0, for instance an AR(1). In the model u_t, the business-cycle factor, is not necessarily linked between ω_t, H_t and i_t. It is only plausible auxiliary assumptions about the technology/endowment process that create those links. It is plausible that when u_t is high ("GDP boom") H_t is high, and ω_t is low. In booms, the conditional intensity of disaster is smaller. Note that this is compatible with the "animal spirits" interpretation of the model.

We can gather the main results from the paper. Stock prices are:

$$P_t = \frac{D_t}{R}\left(1 + \frac{\widehat{H}_t}{R + \phi_H} + \frac{\widehat{g}_t}{R + \phi_g}\right) \text{ with } R = R_* - H_* - g_* \tag{35}$$

$$H_t = p_t\left(B_t^{-\gamma}F_t - 1\right) \tag{36}$$

The price of a bond of maturity T is:

$$Z_t(T) = e^{-(R-H+i_{**})T}\left(1 - \frac{1 - e^{-\phi_i T}}{\phi_i}(i_t - i_{**}) - \frac{\frac{1-e^{-\phi_i T}}{\phi_i} - \frac{1-e^{-\psi_\omega T}}{\psi_i}}{\psi_\omega - \phi_i}\omega_t\right) \tag{37}$$

$$\omega_t = p_t B_t^{-\gamma}F_t i_t^{\#} \tag{38}$$

where i_t is inflation, and ω_t is a bond risk premium.

GDP and Investment are:

$$Y_t = C_t(1 + a + u_t)$$

$$I_t = C_t(a + u_t)$$

Common factors to GDP, bond premia, and stock premia When π_t is smaller, the yield curve is less steep. Hence, we predict that a less steep yield curve predicts future declines in GDP (u_t is high now, and will mean-revert), and also smaller stock market returns. This is consistent with the findings of the literature, in which Cochrane and Piazzesi (2005) find that a low value of their bond-premium factor (which we view π_t as a proxy for) does predict future low stock market returns.

Also, investment is predicted by u_t, while the P/E ratio of the stock market (close to B/M) is a function of H_t and g_t.

Successes and failures of the Q theory of investment The model predicts that the risk-premium $\widehat{H}_t$ has little to do with investment, if it is little correlated with the business cycle, although the slope of the yield curve does predict investment. This is consistent with the poor success of forecasting investment with stock market variables (the Q theory of investment), and the greater success of forecasting it with fixed income variables (Philippon 2006).

9 Conclusion

This paper proposes a unified way to think about a series of puzzles about stocks, bonds and exchange rates. I was surprised by how many finance puzzles could be understood with the lenses of such a

simple model. Given that the model is very simple to state, and to solve (thanks to the modeling "tricks" allowed by linearity-generating processes), it can serve as a simple benchmark for various questions in macroeconomics and finance.

The paper suggests several questions for future research. First of all, it would be crucial to examine empirically the predictions that the model generates, including the relationships between stocks, bond, options and exchange rates.

Second, I presented a way to extend the model to (particular) production economies. A more general extension to production economies would be very useful, as it would constitute a long-sought unification of macroeconomics and finance.

Third, the companion paper (Farhi and Gabaix 2007) suggests that various puzzles in international macroeconomics (including the forward premium puzzle and the excess volatility puzzle on exchange rates) can be accounted for in an international version of the present framework. This gives hope that a unified solution to puzzles in closed-economy and international economics (Obstfeld and Rogoff 2001) may be within reach.

Appendix A. Précis of results on Linearity-Generating processes

The paper constantly uses the Linearity-Generating (LG) processes of Gabaix (2007). This Appendix gathers the main results. LG processes are given by $M_t D_t$, a pricing kernel M_t times a dividend D_t, and X_t, a n-dimensional vector of factors (that can be thought as stationary). For instance, for bonds, the dividend is $D_t = 1$.

Discrete time By definition, a process $M_t D_t (1, X_t)$ is LG if and only if it follows, for all t's:

$$E_t \left[\frac{M_{t+1} D_{t+1}}{M_t D_t} \right] = \alpha + \delta' X_t \tag{39}$$

$$E_t \left[\frac{M_{t+1} D_{t+1}}{M_t D_t} X_{t+1} \right] = \gamma + \Gamma X_t \tag{40}$$

Those conditions write more compactly:

$$E_t Y_{t+1} = \Omega Y_t \text{ with } Y_t = \begin{pmatrix} M_t D_t \\ M_t D_t X_t \end{pmatrix} \text{ and } \Omega = \begin{pmatrix} \alpha & \delta' \\ \gamma & \Gamma \end{pmatrix}$$

Higher moments need not be specified.

The main results that stocks and bonds have simple closed-form expressions. The price of a

stock, $P_t = E_t \left[\sum_{s \geq t} M_s D_s \right] / M_t$, is:

$$P_t/D_t = \frac{1 + \delta' (I_n - \Gamma)^{-1} X_t}{1 - \alpha - \delta' (I_n - \Gamma)^{-1} \gamma} \tag{41}$$

The price-dividend ratio of a "bond", $Z_t (T) = E_t [M_{t+T} D_{t+T}] / (M_t D_t)$, is: (with 0_n a $n-$dimensional row of zeros):

$$Z_t (T) = \begin{pmatrix} 1 & 0_n \end{pmatrix} \cdot \Omega^T \cdot \begin{pmatrix} 1 \\ X_t \end{pmatrix} \tag{42}$$

$$= \alpha^T + \delta' \frac{\alpha^T I_n - \Gamma^T}{\alpha I_n - \Gamma} X_t \text{ when } \gamma = 0 \tag{43}$$

Continuous time In continuous time, $M_t D_t (1, X_t)$ is LG if and if only it follows:

$$E_t \left[\frac{d (M_t D_t)}{M_t D_t} \right] = -\left(a + \beta' X_t \right) dt \tag{44}$$

$$E_t \left[\frac{d (M_t D_t X_t)}{M_t D_t} \right] = -(b + \Phi X_t) dt \tag{45}$$

i.e. more compactly

$$E_t [dY_t] = -\omega Y_t dt \text{ with } Y_t = \begin{pmatrix} M_t D_t \\ M_t D_t X_t \end{pmatrix} \text{ and } \omega = \begin{pmatrix} \alpha & \beta \\ b & \Phi \end{pmatrix}.$$

The price of a stock, $P_t/D_t = E_t \left[\int_t^\infty M_s D_s ds \right] / (M_t D_t)$, is:

$$P_t/D_t = \frac{1 - \beta' \Phi^{-1} X_t}{a - \beta' \Phi^{-1} b} \tag{46}$$

and the price-dividend ratio of a "bond" is: $Z_t (T) = E_t [M_{t+T} D_{t+T}] / (M_t D_t)$

$$Z_t (T) = \begin{pmatrix} 1 & 0_n \end{pmatrix} \cdot \exp \left[-\begin{pmatrix} a & \beta' \\ b & \Phi \end{pmatrix} T \right] \cdot \begin{pmatrix} 1 \\ X_t \end{pmatrix} \tag{47}$$

$$= e^{-aT} + \beta' \frac{e^{-\Phi T} - e^{-aT} I_n}{\Phi - a I_n} X_t \text{ when } b = 0 \tag{48}$$

Appendix B. A background GDP process that does not change the predictions of the endowment economy

This section shows an example of a process for GDP and capital, that does not change any of the asset pricing implications of the model, but allows to talk about the "cyclical" properties of stocks and bonds.

We normalize consumption growth to 0. Hence, consumption is $C_t = C_0 D_t$, where $D_t = \Delta_1...\Delta_t$ the cumulative disaster. $\Delta_t = 1$ is there is no disaster at t, otherwise $\Delta_t = B_t \in (0,1)$.

There is an exogenous labor income "tree" that yields W_t, to be specified soon. There is also a capital stock, K_t. The capital accumulated at the end of period t is K_t. If there is disaster, it shrinks by a factor Δ_t. Then, it yields a rate of return r, and depreciates by δ. GDP is the sum of capital and labor income, and is invested in consumption and investment:

$$Y_t = r\Delta_t K_{t-1} + W_t = C_t + K_t - (1-\delta)\,\Delta_t K_{t-1} \tag{49}$$

We want to design an economy so that the GDP process is:

$$Y_t = C_t \left(1 + a + u_t\right)$$

where u_t indicates the cyclical properties of the economy. We do not specify u_t fully, but it could be an AR(1). $a > 0$ will not matter, but allows various quantities to be non-negative.

For this economy to be in equilibrium, we need to specify the Euler equation, and the GDP equation (49).

The Euler equation gives: $E_t \left[\beta \left(C_{t+1}/C_t\right)^{-\gamma} (1+r)\,\Delta_t\right] = 1$, i.e.

$$(1+r)^{-1} = E\left[\beta \left(C_{t+1}/C_t\right)^{-\gamma} \Delta_{t+1}\right] \equiv \beta \left(1 + p\left(B^{-\gamma} - 1\right)\right)$$

The GDP (49). equation implies,

$$Y_t - C_t = C_t\left(a + u_t\right) = K_t - (1-\delta)\,\Delta_t K_{t-1}$$

hence, with $\rho = 1 - \delta$ assumed to be between 0 and 1,

$$K_t/C_t = (1 - \rho L)^{-1}\left(a + u_t\right) = \frac{a}{1-\rho} + (1 - \rho L)^{-1} u_t$$

Also, we need:

$$
\begin{aligned}
W_t/C_t &= \left(Y_t - r\Delta_t K_{t-1}\right)/C_t = 1 + a + u_t - r\left(\frac{a}{1-\rho} + L\left(1-\rho L\right)^{-1} u_t\right) \\
&= 1 + a\frac{1-\rho-r}{1-\rho} + \left[1 - rL\left(1-\rho L\right)^{-1}\right] u_t \qquad\qquad (50)
\end{aligned}
$$

If we start the economy with $u_t = 0$ for $t \leq 0$, and $K_{-1}/C_{-1} = a/\left(1-\rho\right)$, we have completed our "designer" economy.

Proposition 7 *Define $C_t^* = C_0^* \Delta_1 ... \Delta_t$, and r such that $1 = \beta\left(1+r-\delta\right)E\left[\Delta_t^{1-\gamma}\right]$. Suppose an economy with gross rate of return to capital $\left(1+r\right)\Delta_t$, and an endowment labor income:*

$$
W_t = C_t^*\left(1 + a\frac{1-\rho-r}{1-\rho} + \left[1 - rL\left(1-\rho L\right)^{-1}\right] u_t\right) \qquad\qquad (51)
$$

with $\rho = 1-\delta$, and a representative utility function $\sum \beta^t C_t^{1-\gamma}/\left(1-\gamma\right)$, which is maximized subject to the budget constraint on the GDP, Y_t:

$$
Y_t = r\Delta_t K_{t-1} + W_t = C_t + K_t - \left(1-\delta\right)\Delta_t K_{t-1}
$$

Suppose initially $u_t = 0$ for $t \leq 0$, and $K_{-1} = C_0 a/\left(1-\rho\right)$. Then, the equilibrium process for this economy is:

$$
\begin{aligned}
\text{Consumption} &\quad: \quad C_t = C_t^* \\
\text{GDP} &\quad: \quad Y_t = C_t\left(1 + a + u_t\right) \\
\text{Investment} &\quad: \quad I_t = C_t\left(a + u_t\right)
\end{aligned}
$$

and the capital stock is: $K_t = C_t\left(\frac{a}{1-\rho} + \left(1-\rho L\right)^{-1} u_t\right)$.

We have constructed a "designer" economy, in which (i) the optimal consumption is the same as in the endowment economy (ii) the asset pricing properties are the same as in the endowment economy, but (iii) there business cycle. Hence, we can talk about the cyclical properties of asset prices, where the cycle is indexed by u_t.

Appendix C. Longer Proofs

Proof of Proposition 2 **Continuous time**.

In the proof, I normalize $i_* = 0$. I will show that $M_t D_t (1, i_t, \pi_t)$ is a Linearity-Generating process. I calculate its three signature moments (see Appendix A for the motivation). First,

$$E_t \left[\frac{d(M_t D_t)}{M_t D_t} \right] / dt = \underbrace{-(R + i_t)}_{\text{No disaster term}} + p_t \underbrace{\left(FB_t^{-\gamma} - 1 \right)}_{\text{Disaster term}} = -R + H - i_t$$

Next,

$$E_t \left[\frac{d(M_t D_t i_t)}{M_t D_t} \right] / dt = \underbrace{-(R + i_t) i_t + E_t \left[di_t/dt \mid \text{No disaster at } t \right]}_{\text{No disaster term}} + p_t \underbrace{\left(B_t^{-\gamma} F_t E_t \left[i_{t+} \right] - i_t \right)}_{\text{Disaster term}}$$

$$= -(R + i_t) i_t - (\phi - i_t) i_t + p_t \left(B_t^{-\gamma} F_t \left(i_t + b + i_t^\# \right) - i_t \right)$$

$$= p_t B_t^{-\gamma} F_t b - (R + \phi - H) i_t + p_t B_t^{-\gamma} F i_t^\#$$

and finally, in terms of π_t:

$$E_t \frac{d(M_t D_t \pi_t)}{M_t D_t} / dt = \underbrace{-(R + i_t) \pi_t + E_t d\pi_t/dt}_{\text{No disaster term}} + p_t \underbrace{\left(B_t^{-\gamma} F \pi_t - \pi_t \right)}_{\text{Disaster term}} \tag{52}$$

I conclude that $Y_t = M_t D_t (1, i_t, \pi_t)'$ is a Linearity-Generating process.[22]

To solve for the bond price, I define $\widehat{i}_t = i_t - i_{**}$. Then process $M_t D_t \left(1, \widehat{i}_t, \pi_t \right)$ is Linearity-Generating, with generating matrix:

$$\omega_1 = \begin{pmatrix} R - H + i_{**} & 1 & 0 \\ 0 & R - H + i_{**} + \phi_i & -1 \\ 0 & 0 & R - H + \psi_\pi \end{pmatrix}$$

Indeed,

$$E_t \frac{d \left(M_t D_t \widehat{i}_t \right)}{M_t D_t} / dt = E_t \frac{d(M_t D_t i_t)}{M_t D_t} / dt - i_{**} E_t \frac{d(M_t D_t)}{M_t D_t} / dt$$

$$= pB^{-\gamma} b - (R - H + \phi_i) \left(i_{**} + \widehat{i}_t \right) + \pi_t - i_{**} \left[-(R - H + i_{**}) - \widehat{i}_t \right]$$

$$= \kappa (\phi_i - \kappa) - i_{**} (\phi_i - i_{**}) - (R - H + \phi_i - i_{**}) \widehat{i}_t + \pi_t$$

$$= -(R + \phi_i - H - i_{**}) \widehat{i}_t + \pi_t$$

---

[22] $E_t [dY_t] = -\omega Y_t dt$, with the generator is $\omega = \begin{pmatrix} R - H & 1 & 0 \\ -\kappa (\phi_i - \kappa) & R - H + \phi_i & -1 \\ 0 & 0 & R - H + \phi_\pi \end{pmatrix}$.

Applying Theorem 3 in Gabaix (2007), the price of a bond is: $Z_t(T) = (1, 0, 0) \exp(-\omega_1 T) \left(1, \widehat{i}_t, \pi_t\right)'$, hence the announced result.

Discrete time

The proof is as above.

$$E_t \left[\frac{M_{t+1} D_{t+1}}{M_t D_t} \right] = e^{-R} (1 - i_t) \left(1 + p \left(B^{-\gamma} F - 1\right)\right) = e^{-R} (1 + H) \left(1 - i_* - \widehat{i}_t\right)$$

$$E_t \left[\frac{M_{t+1} D_{t+1}}{M_t D_t} \widehat{i}_{t+1} \right] = e^{-R} (1 - i_t) \{ \underbrace{(1 - p) E_t \left[\widehat{i}_{t+1} \mid \text{No disaster at } t+1\right]}_{\text{No disaster term}} + \underbrace{p B^{-\gamma} F E_t \left[\widehat{i}_{t+1} \mid \text{Disaster at } t+1\right]}_{\text{Disaster term}}]$$

$$= e^{-R} (1 - i_t) \frac{1 - i_*}{1 - i_t} \{ \left(1 - p + p B^{-\gamma} F\right) \rho_i \widehat{i}_t + p B^{-\gamma} F j_t \}$$

$$= e^{-R} (1 + H) (1 - i_*) \left(\rho_i \widehat{i}_t + \frac{p B^{-\gamma} F}{1 + H} \left(j_* + \widehat{j}_t\right) \right) \text{ because } H = p \left(B^{-\gamma} F - 1\right)$$

$$= e^{-R} (1 + H) (1 - i_*) \left(\rho_i \widehat{i}_t + (1 - i_*) \kappa (1 - \rho_i - \kappa) + \pi_t \right)$$

using the definitions (14) and (15). Finally,

$$E_t \left[\frac{M_{t+1} D_{t+1}}{M_t D_t} \widehat{j}_{t+1} \right] = E_t \left[\frac{M_{t+1} D_{t+1}}{M_t D_t} \right] E_t \left[\widehat{j}_{t+1}\right] = e^{-R} (1 + H) (1 - i_t) \cdot \frac{1 - i_*}{1 - i_t} \rho_j \widehat{j}_{t+1}$$

$$= e^{-R} (1 + H) (1 - i_*) \rho_j \widehat{j}_{t+1}$$

so that, using that π_t is proportional to $\widehat{j}_t$ (Eq. 14),

$$E_t \left[\frac{M_{t+1} D_{t+1}}{M_t D_t} \widehat{\pi}_{t+1} \right] = e^{-R} (1 + H) (1 - i_*) \rho_j \widehat{\pi}_{t+1}$$

So, $M_t D_t \left(1, \widehat{i}_t, j_t\right)$ is a LG process, with generating matrix:

$$\Omega = e^{-R} (1 + H) (1 - i_*) \begin{pmatrix} 1 & -1/(1 - i_*) & 0 \\ (1 - i_*) \kappa (1 - \rho_i - \kappa) & \rho_i & 1 \\ 0 & 0 & \rho_\pi \end{pmatrix}. \tag{53}$$

Theorem 1 of Gabaix (2007) says that a zero-coupon bond of maturity T has a price $Z_t(T) = (1, 0, 0)' \Omega^T \left(1, \widehat{i}_t, j_t\right)$. Calculating that integral (diagonalizing the matrix by hand, or using a sym-

bolic calculation software) gives:

$$Z_t = \left(e^{-R}(1+H)(1-i_*)\right)^T \left\{(1-\kappa)^T - \frac{(1-\kappa)^T - (\kappa+\rho_i)}{1-2\kappa-\rho}\left(\hat{i}_t - \kappa\right) - \right. \tag{54}$$
$$\left. \left[\frac{\frac{(\kappa+\rho)^T}{\kappa+\rho_i-\rho_\pi} - \frac{(1-\kappa)^T}{1-\kappa-\rho_\pi}}{1-2\kappa-\rho_i} + \frac{\rho_\pi^T}{(\kappa+\rho_i-\rho_\pi)(1-\kappa-\rho_\pi)}\right]\frac{\pi_t}{1-i_*}\right.$$

The above expression is a bit complicated, which is why the continuous-time version may be easier.

Appendix D. Calibrating the variance

Suppose an LG process centered at 0, $dX_t = -(\phi + X_t) X_t dt + \sigma(X_t) dW_t$, where W_t is a standard Brownian motion. Because of economic considerations, the support of the X_t needs to be some $(X_{\min}, X_{\max})$, with $-\phi < X_{\min} < 0 < X_{\max}$. The following variance process makes that possible:

$$\sigma^2(X) = 2K(1 - X/X_{\min})^2(1 - X/X_{\max})^2 \tag{55}$$

with $K > 0$. K is in units of $[\text{Time}]^{-3}$.

The average variance of X is

$$\overline{\sigma}_X^2 = E\left[\sigma^2(X_t)\right] = \int_{X_{\min}}^{X_{\max}} \sigma(X)^2 p(X) dX.$$

where $p(X)$ is the steady state distribution of X_t. It satisfies the Forward Kolmogorov equation, $d\ln p(X)/dX = 2(\phi + X)/\sigma^2(X) - d\ln \sigma^2(X)/dX$.

Numerical simulations shows that the process volatility is fairly well-approximated by: $\overline{\sigma}_X \simeq K^{1/2}\xi$, with $\xi \simeq 1.3$.

Asset prices often require to analyze the standard deviation of expressions like $\ln(1 + aX_t)$. Numerical analysis shows that the Taylor expansion approximation is a good one: Average volatility of: $\ln(1 + aX_t) \simeq aK^{1/2}\xi$, which numerical simulations prove to be a good approximation too.

The standard deviation of X's steady state distribution is: Standard deviation of $X \simeq (K/\phi)^{1/2}$.

For the steady-state distribution to have a "nice" shape (e.g., be unimodal), the following restrictions appear to be useful: $K \leq -0.2 \cdot \phi X_{\min} X_{\max}$ In the calibration of the paper, I take $K = \phi(-X_{\min})^2 0.1$, which leads to an average volatility of X, $\overline{\sigma}_X \simeq 0.40 \cdot \phi^{1/2}(-X_{\min})$, and a standard deviation of X of $0.1^{.5}(-X\min)$.

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