

PRELIMINARY AND INCOMPLETE
Optimal Inattention to the Stock Market in the
Presence of Transactions Costs*

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Abstract

Recurrent intervals of inattention to the stock market are optimal if it is costly to observe the value of the stock market. At times that consumers observe the value of the stock market, they decide whether to transfer funds between a transactions account from which consumption must be financed and an investment portfolio of equity and riskless bonds. Any transfers of funds are subject to a proportional transactions cost, so the consumer may choose not to transfer any funds on a particular observation date. In general, the optimal adjustment rule—including the size and direction of transfers, and the time of the next observation—is state-dependent. Surprisingly, we find that eventually the consumer's optimal behavior evolves to a situation with a purely time-dependent rule, with a constant interval of time between observations. This interval of time can be substantial even with tiny observation costs.

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A pervasive finding in studies of microeconomic choice is that adjustment to economic news tends to be sluggish and infrequent. Individuals rebalance their portfolios and revisit their spending behavior at discrete and sometimes long intervals of time. During these intervals of time, inaction is the rule. Similar sorts of inaction also characterize the financing, investment, and pricing decisions of firms.

These observations have led economists to formulate models that are consistent with this microeconomic behavior. One question of particular interest is whether, and to what extent, infrequent adjustment at the micro level can help account for certain macroeconomic outcomes. For instance, if firms adjust their prices infrequently, then monetary policy could have important short-run real effects. Similarly, if individuals take several months or even years to adjust their portfolios and their spending plans, the standard predictions of the consumption smoothing and portfolio choice theories might fail.

Formal models of infrequent adjustment are often described as either time-dependent or state-dependent. In models with time-dependent adjustment, adjustment is triggered simply by calendar time and is independent of the state of the economy. In models with state-dependent adjustment, adjustment takes place only when the state of the economy reaches some trigger value, so the timing of adjustments is endogenous. A classic type of state-dependent adjustment is the (S,s) model. The difference between time-dependent and state-dependent models can have crucial implications for important economic questions. For instance, monetary policy has substantial real effects that persist for several quarters if firms change their prices according to a time-dependent rule. However, if firms adjust their prices according to a state-dependent rule, then monetary policy may have little or no effect on the real economy. (See e.g. Caplin and Spulber (1987) and Golosov and Lucas (2007).)

In this paper we develop and analyze an optimizing model that can generate both time-dependent adjustment and state-dependent adjustment. The economic context is an infinite-horizon continuous-time model of consumption and portfolio choice that builds on the framework of Merton (1971). We augment Merton's model by requiring that consumption can be purchased only with a liquid asset and by introducing two sorts of costs—a cost of observing the price of equity and a cost of transferring assets between the investment portfolio consisting of risky equity and riskless bonds and a transactions account consisting of liquid assets. Because it is costly for the consumer to observe the value of risky equity, the consumer will choose to observe this value only at discretely-spaced points in time. At these observation times, the consumer will choose when next to observe the value of equity, and will also execute any transfers between the investment portfolio and the transactions account, as well

as choose the path of consumption until the next observation date. In general, the timing of asset transfers will be state-dependent. The relevant state of the consumer's balance sheet is captured by x_t , defined as the ratio of the balance in the transactions account to the contemporaneous value of the investment portfolio. On any given observation date, the consumer chooses the date of the next observation, but, depending on the value of x_t that is realized on the next observation date, the consumer may or may not transfer assets between the investment portfolio and the transactions account on that date. Because the timing of asset transfers depends on the value of x_t , we describe these transfers as state-dependent. A surprising result of our analysis, however, is that eventually the consumer's holdings in the investment portfolio and the transactions account will evolve to a situation in which the optimal timing of asset transfers is purely time-dependent. Indeed, when the asset holdings get to this stage, the optimal time between successive asset transfers is constant.

We will show that optimal behavior can be described by three intervals for the value of x_t . When x_t on an observation date has a high value, so that the consumer's balance sheet is relatively heavy in the transactions account, the consumer will find it optimal to use some of the transactions account to purchase additional assets in the investment portfolio. Alternatively, when x_t has an intermediate value on an observation date, the consumer will not find it worthwhile to pay the transactions costs associated with either transferring assets into the investment portfolio or transferring assets out of the investment portfolio. This is the inaction situation that makes the timing of asset transfers state-dependent. Finally, when x_t is low on an observation date, the consumer will sell some assets from the investment portfolio to replenish the transactions account in order to finance consumption until the next observation date.

We show that eventually, though not necessarily immediately, optimal behavior will lead to a low value of x_t on an observation date. Once a low value of x_t is realized on an observation date, the consumer will not transfer more assets to the transactions account than are needed to finance consumption until the next observation date. The reason is that it is both costly to transfer assets and the liquid asset in the transactions account earns a lower riskless rate of return than the riskless bond in the investment portfolio. In this case, the consumer will plan to hold a zero balance in the transactions account on the next observation date, so that x_t will equal zero on the next observation date. Thus, on the next observation date, x_t will have a low value and the situation repeats itself: x_t will equal zero on every subsequent observation date and the optimal interval between successive observations will be constant, which is a purely time-dependent rule.

This paper relates to two strands of the literature. The first strand is the large literature on transactions costs. In the models by Baumol (1952) and Tobin (1956), which are the forerunners of the cash-in-advance model used in macroeconomics, consumers can hold two riskless assets that pay different rates of return—money, which pays zero interest, and a riskless bond that pays a positive rate of interest. Consumers are willing to hold money, despite the fact that its rate of return is dominated by the rate of return on riskless bonds because goods have to be purchased with money. That is, money offers liquidity services. In our model, the liquid asset could pay a zero rate of return (as is the case for currency when the rate of inflation is zero, so the real return on currency is zero) or it could pay a positive rate of return as is the case for some liquid deposits in M1 and M2. But, as in the Baumol-Tobin literature, the rate of return on the liquid asset is lower than the rate of return on riskless bonds in the investment portfolio.

A more recent literature on portfolio transactions costs, including Constantinides (1986) and Davis and Norman (1990) models the costs of transferring assets between stocks and bonds in the investment portfolio as proportional to the size of the transfers. Here we also model transactions costs as proportional to the size of the transactions, but the transactions cost apply only to transfers between the liquid asset in the transactions account on the one hand and the investment portfolio of stocks and bonds on the other. We do not model the costs of reallocating stocks and bonds *within* the investment portfolio. For a retired consumer who finances consumption by withdrawing assets from a tax-deferred retirement account, the transactions cost of withdrawing the assets includes the taxes paid at the time of withdrawal. The marginal tax rate for most consumers in this situation, which is part of the transactions cost of transferring assets from the investment portfolio to the transactions account, is likely to be far greater than any transactions costs associated with reallocating the stocks and bonds within the investment portfolio.

A second strand of the literature, which includes Abel, Eberly, and Panageas (2007), Duffie and Sun (1990) and Gabaix and Laibson (2002) analyzes infrequent adjustment of portfolios that arises because it is costly to process information.¹ If these costs are proportional to the value of the entire investment portfolio, then in a continuous-time framework, the consumer will choose not to observe the value of the investment portfolio continuously. The two closest antecedents to our current paper are Duffie and Sun (1990) and Abel, Eberly,

¹Reis (2006) develops and analyzes a model of optimal inattention for a consumer who faces a cost of observing additive income, such as labor income. In this case, the consumer will adjust consumption to changes in income only at discretely-spaced points in time. In this model, the consumer can hold only a single riskless asset so there is no portfolio allocation problem.

and Panageas (2007). In our current paper, we show that the consumer’s behavior differs in three different intervals of the state variable x_t . In particular, we show that when x_t is low, the consumer will plan to arrive at the next observation date with a zero balance in the transactions account, and that the length of time between subsequent observations is constant. Duffie and Sun derive this result, but they implicitly confined attention to low values of x_t . Here we show that behavior is potentially different for intermediate and high values of x_t , which are situations not considered by Duffie and Sun. But we go on to show that eventually x_t will indeed become low on an observation date and then will remain low on all subsequent observation dates. In this sense, Duffie and Sun confine attention to the long run and we consider the transition path to the long run as well as the long run. Importantly, the consideration of behavior outside of the long-run situation allows the model to incorporate state-dependent adjustment as well as purely time-dependent adjustment. A second contribution of this paper relative to Duffie and Sun is that we offer an assessment of the length of the interval of time between consecutive observations in the long run. This assessment includes an analytic component and a quantitative component based on a quadratic approximation. Finally, relative to our own earlier paper, this paper explicitly allows separate consideration of observation costs and transactions costs. In addition, our earlier paper assumes that the investment portfolio is continuously re-balanced by a portfolio manager who charges a fee proportional to the size of the portfolio (and thus the fee is not separately identifiable from an observation cost that is proportional to the size of the portfolio) whereas the current paper does not allow re-balancing of the investment portfolio between observation dates. We will see that whether the portfolio is re-balanced continuously or not can affect the optimal interval between observations by a factor of about two.

We set up the consumer’s decision problem in Section 1. The consumer lives in continuous time but observes the value of the investment portfolio and makes decisions—about consumption, transfers between the investment portfolio and the transactions account, the share of the investment portfolio to hold in risky equity, and the next date at which to observe the value of the investment portfolio—at discretely spaced points in time. In Section 2, we analyze the optimal path of consumption over the discrete interval of time until the next observation date. Then we analyze the optimal transfers between the investment portfolio and the transactions account in Section 3. To analyze these optimal transfers will we also develop properties of the value function. Next, in Section 4, we derive conditions that characterize the optimal allocation of the investment portfolio, and we exploit these condi-

tions in Section 5 to determine when the consumer will have exhausted the liquid assets in the transactions account when arriving at an observation date. These results serve as the basis for the graphical illustration of the dynamic behavior of the transaction account and the investment portfolio in Section 6. Then in Section 7 we show that once the consumer arrives at an observation date with a zero balance in the transactions account, she will arrive at *all* subsequent observation dates with zero liquid assets. In formal terms, the amount of liquid assets held when arriving at an observation date is a stochastic process, and zero is an absorbing state for this process. We show in Section 7 that the absorbing state is reached in finite time. Of course, the consumer does not continuously hold zero liquid assets in the absorbing state. On each observation date, the consumer will sell assets from the investment portfolio to replenish the liquid assets in the transactions account, and then will gradually but completely deplete the transaction account to finance consumption until the next observation date. In the absorbing state, the time interval between observation dates is constant and we analyze this length of this interval in Section 8.

1 Consumer's Decision Problem

Consider an infinitely-lived consumer whose objective at time t is to maximize

$$E_t \left\{ \int_0^\infty \frac{1}{1-\alpha} c_{t+s}^{1-\alpha} e^{-\rho s} ds \right\}, \quad (1)$$

where the coefficient of relative risk aversion is $0 < \alpha \neq 1$ and the rate of time preference is $\rho > 0$. The consumer does not earn any labor income but has wealth that consists of risky equity, riskless bonds, and a liquid asset used for transactions. Risky equity and riskless bonds are held in an investment portfolio and cannot be used directly to purchase consumption. Consumption must be purchased with the liquid asset, which the consumer holds in a transactions account.

Equity is a non-dividend-paying stock with a price P_t that evolves according to a geometric Brownian motion

$$\frac{dP_t}{P_t} = \mu dt + \sigma dz, \quad (2)$$

where $\mu > 0$ is the mean rate of return and σ is the instantaneous standard deviation. The riskless bond has a constant instantaneous rate of return r_f that is positive and less than the mean rate of return on equity, so $0 < r_f < \mu$. The total value of the investment portfolio,

consisting of equity and riskless bonds, is S_t at time t .

At time t , the consumer holds X_t in the liquid asset, which pays a riskless instantaneous rate of return r_L , where $0 \leq r_L < r_f$. The rate of return on the liquid asset, r_L , is lower than the rate of return on the riskless bond in the investment portfolio, r_f , because the liquid asset provides transactions services not provided by the bond in the investment portfolio.

We introduce two types of costs into the consumer's intertemporal decision problem: a cost to observe the value of the investment portfolio and a cost to transfer assets between the investment portfolio and the transactions account. The investment portfolio comprises the consumer's holding of equity and riskless bonds. The value of the riskless bonds is known to the consumer because its value evolves deterministically. However, the consumer can observe the value of risky equity held in his portfolio only by paying a fraction θ , $0 < \theta < 1$, of the contemporaneous value of the equity. As a result of this observation cost, the consumer will choose not to observe the value of the investment portfolio continuously. Instead, the consumer will optimally choose discretely-spaced points in time, t_j , $j = 0, 1, 2, \dots$, at which to observe the value of equity and hence S_t ; during the periods of time between consecutive observation dates, the consumer will be inattentive to the value of equity and S_t .

Suppose the consumer observes the value of the investment portfolio at time t_j and next observes its value at time $t_{j+1} = t_j + \tau_j$. Immediately upon observing the value of S_{t_j} , the consumer may transfer assets between the investment portfolio and the liquid asset in the transactions account (at a cost described below) so that at time t_j^+ the value of the investment portfolio is $S_{t_j^+}$. The consumer chooses to hold a fraction ϕ_j of $S_{t_j^+}$ as risky equity and fraction $1 - \phi_j$ in riskless bonds and does not rebalance the investment portfolio before the next observation date.² When the consumer next observes the value of the investment portfolio, at time $t_j + \tau_j$, its value, *after paying the observation cost*, is

$$S_{t_j+\tau_j} = (1 - \theta) \phi_j \frac{P_{t_j+\tau_j}}{P_{t_j}} S_{t_j^+} + (1 - \phi_j) (R^f)^{\tau_j} S_{t_j^+}, \quad (3)$$

where $R^f \equiv e^r$ is the gross rate of return on the riskless bond per unit of time. Since $r_f > r_L$, we have $R^f > R^L \equiv e^{r_L}$, which is the gross rate of return on the liquid asset per unit of time.³

²The consumer does not learn any new information between time t_j^+ and time t_{j+1} and hence cannot adjust the portfolio in response to any news that arrives during this interval of inattention. It is possible that the consumer could decide at time t_j^+ to exchange equity for bonds at some time(s) before t_{j+1} , but we do not consider this possibility in this paper.

³We have assumed that the consumer can costlessly observe the values of the riskless bond in the invest-

Define

$$R(t_j, t_j + \tau_j) \equiv (1 - \theta) \phi_j \frac{P_{t_j + \tau_j}}{P_{t_j}} + (1 - \phi_j) (R^f)^{\tau_j} \quad (4)$$

and observe that

$$S_{t_j + \tau_j} = R(t_j, t_j + \tau_j) S_{t_j}^+ \quad (5)$$

Thus, $R(t_j, t_j + \tau_j)$ is the gross rate of return on the investment portfolio from time t_j^+ to time $t_j + \tau_j$, net of the observation cost at time $t_j + \tau_j$.

The second cost is a transaction cost that the consumer must pay whenever transferring assets between the investment portfolio and the liquid asset. When the consumer uses some of the liquid asset to purchase additional assets in the investment portfolio, the transaction cost is $\psi_b \Delta S \geq 0$, where $\Delta S \geq 0$ is the increase in the size of the investment portfolio and $\psi_b \geq 0$ is a proportional transactions cost. When the consumer sells some of the investment portfolio to increase the amount held in the liquid asset, the transaction cost is $-\psi_s \Delta S \geq 0$, where $-\Delta S \geq 0$ is the size of the decrease in the investment portfolio and $0 \leq \psi_s < 1$ is a proportional transactions cost. We allow, but do not require, ψ_s and ψ_b to be equal. Perhaps the most obvious interpretation of the proportional transactions costs, ψ_s and ψ_b , is that they represent brokerage fees. Another interpretation presents itself if we consider the investment portfolio to be a tax-deferred account such as a 401k account. In this case, the consumer must pay a tax on withdrawals from the investment portfolio, and ψ_s would be the consumer's income tax rate, which would be substantially higher than a brokerage fee.⁴

Suppose that we are at time 0 and the consumer has just observed the value of the investment portfolio, S_0 . The next observation of the investment portfolio will be at date

ment portfolio and the riskless liquid asset in the transactions account. However, we can allow for a cost of θ_f per unit value of the riskless bond to observe its value and a cost of θ_L per unit value of the liquid asset to observe its value. We can incorporate these observation costs by defining $(R^f)^\tau \equiv (1 - \theta_f) e^{r_f \tau}$ and $(R^L)^\tau \equiv (1 - \theta_L) e^{r_L \tau}$, provided that $(R^f)^\tau > (R^L)^\tau$, which would be true, for example, if $\theta_f = \theta_L$. Including these observation costs allows one to model the observation cost as being proportional to total wealth simply by setting $\theta_f = \theta_L = \theta$. A rationale for this formulation is that the cost is a leisure cost, which is proportional to total wealth. For instance, if the instantaneous utility function is $u(c, l) = \frac{1}{1-\alpha} c^{1-\alpha} f(l)$, where l is leisure, then the marginal value of leisure is $u_l(c, l) = \frac{1}{1-\alpha} c^{1-\alpha} f'(l)$. Thus, for a given level of leisure, the cost of a unit of foregone leisure to observe the value of assets is proportional to $c^{1-\alpha}$, in utility terms, which is proportional to c in terms of real goods, which is proportional to wealth in terms of real goods.

⁴This interpretation of ψ_s as a tax rate is most plausible if the consumer only withdraws money from the investment portfolio and never transfers assets into the investment portfolio. As we will see in Section 7, the long-run is characterized by precisely this situation in which the consumer never transfers funds into the investment portfolio.

τ . The length of time between adjacent observations need not be constant over time. In general, the optimal length of time until the next observation date is a function of the state variables, X and S . For an observation date, t_j , the optimal value of the next observation date is $t_j + \hat{\tau}(X_{t_j}, S_{t_j})$.

Immediately after observing X_0 and S_0 at time 0, the consumer transfers assets between the transactions account and the investment portfolio. If the consumer liquidates some assets in the investment portfolio, thereby reducing the size of the investment portfolio to S_{0+} from S_0 , the balance in the transactions account increases to X_{0+} from X_0 . Taking account of the transactions cost ψ_s , we have

$$X_{0+} = X_0 + (1 - \psi_s)(S_0 - S_{0+}), \text{ if } S_0 \geq S_{0+}. \quad (6)$$

If the consumer uses some of the liquid asset to increase the size of the investment portfolio to S_{0+} from S_0 , the holding of the liquid asset falls to X_{0+} from X_0 . Taking account of the transactions cost ψ_b , we have

$$X_{0+} = X_0 + (1 + \psi_b)(S_0 - S_{0+}), \text{ if } S_0 \leq S_{0+}. \quad (7)$$

2 Consumption until the Next Observation Date

Let C be the present value, discounted at the rate r_L , of the flow of consumption over the interval of time from 0^+ to τ . Specifically

$$C = \int_0^\tau c_s e^{-r_L s} ds, \quad (8)$$

where the path of consumption c_s , $0^+ < c_s \leq \tau$, is chosen to maximize the discounted value of utility over the interval from 0^+ to τ . Let

$$U(C; \tau) = \max_{\{c_s\}_{s=0^+}^\tau} \int_0^\tau \frac{1}{1 - \alpha} c_s^{1-\alpha} e^{-\rho s} ds \quad (9)$$

subject to a given value of C in equation (8). Since the consumer does not learn any new information between time 0^+ and time τ , the maximization in equation (9) is a standard intertemporal optimization under certainty. The optimal values of consumption during this interval of time satisfy the condition that the product of the intertemporal marginal rate

of substitution between times 0^+ and s , $\left(\frac{c_s}{c_{0^+}}\right)^{-\alpha} e^{-\rho s}$, and the gross rate of return between those times, $e^{r_L s}$, equals one, so that

$$c_s = e^{-\frac{\rho-r_L}{\alpha}s} c_{0^+}, \text{ for } 0^+ \leq s \leq \tau. \quad (10)$$

Substituting c_s from equation (10) into equation (8) yields

$$C = h(\tau) c_{0^+} \quad (11)$$

where

$$h(\tau) \equiv \int_0^\tau e^{-\omega s} ds = \frac{1 - e^{-\omega\tau}}{\omega} \quad (12)$$

and we assume that

$$\omega \equiv \frac{\rho - (1 - \alpha)r_L}{\alpha} > 0. \quad (13)$$

Equations (10) and (11) imply that

$$c_s = [h(\tau)]^{-1} e^{-\frac{\rho-r_L}{\alpha}s} C, \quad \text{for } 0 < s \leq \tau. \quad (14)$$

Substituting equation (14) into equation (9), and using the definition of $h(\tau)$ in equation (12) yields

$$U(C; \tau) = \frac{1}{1 - \alpha} [h(\tau)]^\alpha C^{1-\alpha}. \quad (15)$$

Since all of the consumption during the interval of time from 0^+ to τ is financed from the liquid asset in the transactions account, which earns an instantaneous riskless rate of return r_L , we have

$$X_\tau = (X_{0^+} - C) (R^L)^\tau. \quad (16)$$

3 The Value Function and Its Properties

The value function $V(X_{t_j}, S_{t_j})$ is the maximized value of the consumer's utility from the observation date t_j onward, with the consumer choosing the optimal length of time until the next observation date $\tau_j \equiv \hat{\tau}(X_{t_j}, S_{t_j})$, the optimal present value of the stream of consumption until the next observation date, C , the optimal values of $X_{t_j^+}$ and $S_{t_j^+}$ and the

share of the investment portfolio held in risky equity at time t_j , ϕ_j . Therefore,

$$V(X_{t_j}, S_{t_j}) = \max_{C, X_{t_j^+}, S_{t_j^+}, \phi_j, \tau_j} \frac{1}{1 - \alpha} [h(\tau_0)]^\alpha C^{1-\alpha} + \beta^\tau E_0 \{V(X_{t_j+\tau_j}, S_{t_j+\tau_j})\}. \quad (17)$$

The value function defined in equation (17) is homogeneous of degree $1 - \alpha$ in X_{t_j} and S_{t_j} so that

$$V(X_{t_j}, S_{t_j}) = \frac{1}{1 - \alpha} S_{t_j}^{1-\alpha} v(x_{t_j}) \quad (18)$$

where

$$x_t \equiv \frac{X_t}{S_t}. \quad (19)$$

The maximization on the right hand side of equation (17) involves the choice of five variables at time t_j^+ : C , $X_{t_j^+}$, $S_{t_j^+}$, ϕ_j , and τ_j . Our strategy for analyzing this maximization is to begin by finding the optimal values of some of these five variables given arbitrary values of the remaining variables. To implement this strategy we introduce two constructs that we define formally in this section: the restricted value function $F(X_0, S_0; \tau)$ and the conditional value function $\widehat{V}(X_{t_j}, S_{t_j}; \tau_j)$.

3.1 The Restricted Value Function $F(X_0, S_0; \tau)$

Suppose that time 0 is an observation date and define the restricted value function $F(X_0, S_0; \tau)$ as the maximized expected present value of the consumer's infinite-horizon utility from date 0 onward *given that the consumer will not transfer any assets between the investment portfolio and the transactions account until time τ* (so that $X_{0^+} = X_0$ and $S_{0^+} = S_0$). Specifically

$$F(X_0, S_0; \tau) = \max_{C, \phi_0} U(C; \tau) + e^{-\rho\tau} E \{V(X_\tau, S_\tau)\} \quad (20)$$

where

$$X_\tau = (X_0 - C) (R^L)^\tau \quad (21)$$

and

$$S_\tau = R(0, \tau) S_0. \quad (22)$$

$F(X_0, S_0; \tau)$ is strictly increasing in X_0 and S_0 and is homogeneous of degree $1 - \alpha$ in X_0 and S_0 . It is straightforward to show that the weak concavity of $V(X_{t_j}, S_{t_j})$ and the strong concavity of $U(C)$ imply that $F(X_0, S_0; \tau)$ is strictly concave in X_0 and S_0 .

Lemma 1 : $V(X, S)$ is concave, and $F(X_0, S_0; \tau)$ is strictly concave in X_0 and S_0 .

Proof. See Appendix. ■

The function $F(X, S; \tau)$ is homogeneous of degree $1 - \alpha$ in X and S , and hence can be written as

$$F(X, S; \tau) = \frac{1}{1 - \alpha} S^{1 - \alpha} f(x; \tau). \quad (23)$$

Define the function

$$m_f(x; \tau) \equiv (1 - \alpha) \frac{f(x; \tau)}{f'(x; \tau)} - x. \quad (24)$$

It can be shown that⁵

$$m_f(x; \tau) = \frac{F_S(X, S; \tau)}{F_X(X, S; \tau)} > 0. \quad (25)$$

Thus, $m_f(x; \tau)$ is the (negative of the) slope of the level set $F(X, S; \tau) = \bar{F}$, i.e., it is $-\frac{dX}{dS}|_{F(X, S; \tau) = \bar{F}}$ and can be interpreted as the marginal rate of substitution between X and S . The strict concavity of $F(X, S; \tau)$ implies that $m'_f(x; \tau) > 0$.⁶

⁵Differentiate $F(X, S; \tau)$ with respect to X and S , respectively, and use the fact that $F(X, S; \tau)$ is strictly increasing to obtain $F_X(X, S; \tau) = \frac{1}{1 - \alpha} S^{-\alpha} f'(x; \tau) > 0$ and $F_S(X, S; \tau) = \left[f(x; \tau) - \frac{1}{1 - \alpha} x f'(x; \tau) \right] S^{-\alpha} > 0$. Divide the expression for $F_S(X, S; \tau)$ by the expression for $F_X(X, S; \tau)$ to obtain equation (25) in the text.

⁶Differentiating the expressions for $F_X(X, S; \tau)$ and $F_S(X, S; \tau)$ in footnote 5 with respect to X and S yields

$$F_{XX}(X, S; \tau) = \frac{1}{1 - \alpha} S^{-\alpha - 1} f''(x; \tau) \quad (F1)$$

$$F_{XS}(X, S; \tau) = -[\alpha f'(x; \tau) + x f''(x; \tau)] \frac{1}{1 - \alpha} S^{-\alpha - 1}. \quad (F2)$$

and

$$F_{SS}(X, S; \tau) = [2\alpha x f'(x; \tau) + x^2 f''(x; \tau) - \alpha(1 - \alpha) f(x; \tau)] \frac{1}{1 - \alpha} S^{-\alpha - 1}. \quad (F3)$$

Define

$$H(X, S; \tau) \equiv [F_{XX}(X, S; \tau)] [F_{SS}(X, S; \tau)] - [F_{XS}(X, S; \tau)]^2. \quad (F4)$$

Substitute equations (F1), (F2), and (F3) into equation (F4) and simplify to obtain

$$H(X, S; \tau) = \left(-(1 - \alpha) \frac{f(x; \tau) f''(x; \tau)}{[f'(x; \tau)]^2} - \alpha \right) \left(\frac{1}{1 - \alpha} \right)^2 \left(\frac{1}{f'(x; \tau)} \right)^2 \alpha S^{-2(\alpha + 1)}. \quad (F5)$$

Differentiate the expression for the marginal rate of substitution, $m_f(x; \tau)$, in equation (24) to obtain

$$m'_f(x; \tau) = -\alpha - (1 - \alpha) \frac{f(x; \tau) f''(x; \tau)}{[f'(x; \tau)]^2}. \quad (F6)$$

3.2 The Conditional Value Function $\widehat{V}(X_0, S_0; \tau)$

Suppose that time 0 is an observation date and define the conditional value function $\widehat{V}(X_{t_j}, S_{t_j}; \tau_j)$ as the maximized value of the consumer's expected present value of lifetime utility from observation date 0 onward, for an arbitrary given value of τ , the next observation date, and the optimal values all subsequent observation. Thus, at observation date 0,

$$\widehat{V}(X_0, S_0; \tau) = \max_{C, X_{0+}, S_{0+}, \phi_0} \frac{1}{1-\alpha} [h(\tau_0)]^\alpha C^{1-\alpha} + \beta^\tau E_0 \{V(X_\tau, S_\tau)\} \quad (26)$$

and

$$V(X_0, S_0) = \max_{\tau_0} \widehat{V}(X_0, S_0; \tau_0) = \widehat{V}(X_0, S_0; \widehat{\tau}_0(X_0, S_0)). \quad (27)$$

Since $\widehat{V}(X_{t_j}, S_{t_j}; \tau_j)$ is homogenous of degree $1-\alpha$ in (X_{t_j}, S_{t_j}) we have

$$\widehat{V}(X_{t_j}, S_{t_j}; \tau_j) = \frac{1}{1-\alpha} S_{t_j}^{1-\alpha} \widehat{v}(x_{t_j}; \tau_j), \quad (28)$$

and

$$\widehat{v}(x_{t_j}; \tau_j) \equiv \widehat{V}(x_{t_j}, 1; \tau_j) \quad (29)$$

where $x_t \equiv \frac{X_t}{S_t}$.

Use the definition of $F(X_0, S_0; \tau)$ in equation (20) and the definition of $\widehat{V}(X_0, S_0; \tau)$ in equation (26) to obtain

$$\widehat{V}(X_0, S_0; \tau) = \max_{X_{0+}, S_{0+}} F(X_{0+}, S_{0+}; \tau) \quad (30)$$

subject to equations (6) and (7). Substituting equations (6) and (7) into equation (30)

Use equation (F6) to rewrite equation (F5) as

$$H(X, S; \tau) = \left(\frac{1}{1-\alpha}\right)^2 \left(\frac{1}{f'(x; \tau)}\right)^2 \alpha S^{-2(\alpha+1)} m'_f(x; \tau) \quad (F7)$$

Strict concavity of $F(X, S; \tau)$ implies that $H(X, S; \tau) > 0$, so equation (F7) implies that

$$m'_f(x; \tau) > 0 \quad (F8)$$

yields

$$\widehat{V}(X_0, S_0; \tau) = \max \left[\begin{array}{l} \max_{S_{0+} \leq S_0} F(X_0 + (1 - \psi_s)(S_0 - S_{0+}), S_{0+}; \tau), \\ \max_{S_{0+} \geq S_0} F(X_0 + (1 + \psi_b)(S_0 - S_{0+}), S_{0+}; \tau) \end{array} \right] \quad (31)$$

The right hand side of equation (31) contains two maximization problems. The first-order conditions for the first maximization problem are

$$\frac{F_S(X_{0+}, S_{0+}; \tau)}{F_X(X_{0+}, S_{0+}; \tau)} - (1 - \psi_s) \geq 0 \quad (32)$$

and

$$\left[\frac{F_S(X_{0+}, S_{0+})}{F_X(X_{0+}, S_{0+})} - (1 - \psi_s) \right] [S_0 - S_{0+}] = 0. \quad (33)$$

Use equation (25) to rewrite equation (32) as

$$m_f(x_{0+}; \tau) \geq 1 - \psi_s. \quad (34)$$

Define $\pi_1(\tau)$ as the unique positive value of x that satisfies

$$m_f(\pi_1(\tau); \tau) = 1 - \psi_s. \quad (35)$$

If, and only if, $S_{0+} < S_0$, the consumer sells some assets in the investment portfolio at time 0 and uses the proceeds, net of transactions cost, to increase the transactions account, so $X_{0+} > X_0$. Therefore, $S_{0+} < S_0$ if and only if $x_{0+} \equiv \frac{X_{0+}}{S_{0+}} > \frac{X_0}{S_0} \equiv x_0$, so equation (33) can be rewritten as

$$[m_f(x_{0+}; \tau) - (1 - \psi_s)] [x_0 - x_{0+}] = 0. \quad (36)$$

Equations (34), (35), and (36) imply⁷ that if $x_0 < \pi_1(\tau)$, then $x_{0+} = \pi_1(\tau)$, so that at time 0⁺ the consumer sells some assets in the investment portfolio and increases the holding of liquid assets in the transactions account. If $x_0 \geq \pi_1(\tau)$, then the investor does not sell any assets from the investment portfolio.

Now consider the second maximization problem on the right hand side of equation (31), $\max_{S_{0+} \geq S_0} F(X_0 + (1 + \psi_b)(S_0 - S_{0+}), S_{0+}; \tau)$. The first-order conditions for this maxi-

⁷If $x_0 < \pi_1(\tau)$, then $m_f(x_0; \tau) < m_f(\pi_1(\tau); \tau) = 1 - \psi_s \leq m_f(x_{0+}; \tau)$, so $x_0 < x_{0+}$. Therefore, equation (36) implies that $m_f(x_{0+}; \tau) = 1 - \psi_s = m_f(\pi_1(\tau); \tau)$, which implies that $x_{0+} = \pi_1(\tau)$.

mization problem are

$$m_f(x_{0+}; \tau) \leq (1 + \psi_b) \quad (37)$$

and

$$[m_f(x_{0+}; \tau) - (1 + \psi_s)] [S_0 - S_{0+}] = 0. \quad (38)$$

Define $\pi_2(\tau) \geq \pi_1(\tau)$ as the unique positive value of x that satisfies

$$m_f(\pi_2(\tau); \tau) = 1 + \psi_b. \quad (39)$$

If, and only if, $S_{0+} > S_0$, the consumer buys some assets for the investment portfolio at time 0^+ by using some liquid assets and thereby reducing the size of the transactions account so $X_{0+} < X_0$. Therefore, $S_{0+} > S_0$ if and only if $x_{0+} \equiv \frac{X_{0+}}{S_{0+}} < \frac{X_0}{S_0} \equiv x_0$, so equation (38) can be rewritten as

$$[m_f(x_{0+}; \tau) - (1 + \psi_b)] [x_0 - x_{0+}] = 0. \quad (40)$$

Equations (37), (39), and (40) imply⁸ that if $x_0 > \pi_2(\tau)$, then $x_{0+} = \pi_2(\tau)$, so the consumer uses some of the liquid asset in the transactions account to buy assets in the investment portfolio. If $x_0 \leq \pi_2(\tau)$, then the investor does not transfer any assets into the investment portfolio.

We can easily summarize the transactions between the investment portfolio and the transactions account at time 0^+ as follows: If $x_0 < \pi_1(\tau)$, the consumer sells enough assets from the investment portfolio to increase x_{0+} to $\pi_1(\tau)$. If $\pi_1(\tau) \leq x_0 \leq \pi_2(\tau)$, then the consumer does not transfer any assets between the investment portfolio and the transactions account at time 0^+ . If $x_0 > \pi_2(\tau)$, the consumer buys enough assets at time 0^+ to add to the investment portfolio to decrease x_{0+} to $\pi_2(\tau)$.

To describe the indifference curves of the conditional value function $\widehat{V}(X_0, S_0; \tau)$, it will be useful to define

$$m_{\widehat{v}}(x; \tau) \equiv (1 - \alpha) \frac{\widehat{v}(x; \tau)}{\widehat{v}'(x; \tau)} - x, \quad (41)$$

which is the analog of $m_f(x; \tau)$ defined in equation (24). Just as $m_f(x; \tau)$ is the (negative of the) slope of the level set $F(X, S; \tau) = \overline{F}$, $m_{\widehat{v}}(x; \tau)$ is the (negative of the) slope of the level set $\widehat{V}(X, S; \tau) = \overline{V}$. Thus, $m_{\widehat{v}}(x; \tau)$ is the marginal rate of substitution between X

⁸If $x_0 > \pi_2$, then $m_f(x_0; \tau) > m_f(\pi_2; \tau) = 1 + \psi_b \geq m_f(x_{0+}; \tau)$, so $x_0 > x_{0+}$. Therefore, equation (40) implies that $m_f(x_{0+}; \tau) = 1 + \psi_b = m_f(\pi_2; \tau)$, which implies that $x_{0+} = \pi_2$.

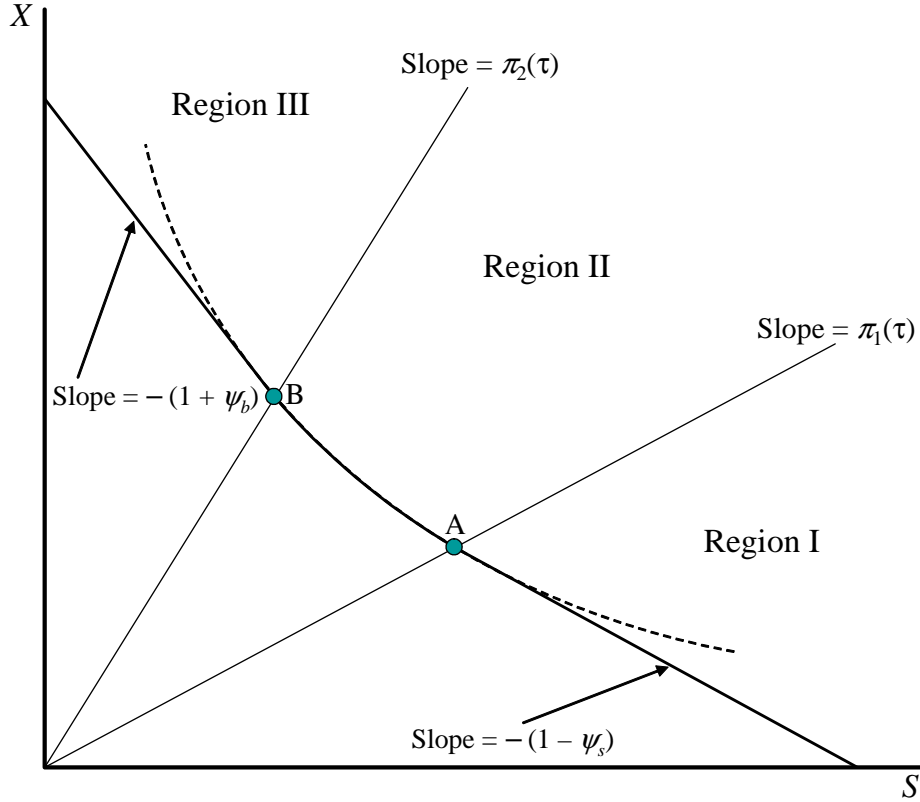


FIGURE 1:

and S for the value function $\widehat{V}(X, S; \tau)$, that is

$$m_{\widehat{v}}(x; \tau) = \frac{\widehat{V}_S(xS, S; \tau)}{\widehat{V}_X(xS, S; \tau)}. \quad (42)$$

Since the value function is weakly concave, rather than strictly concave, in X and S , $m'_{\widehat{v}}(x; \tau) \geq 0$. As we will show, there are some intervals of x for which $m'_{\widehat{v}}(x; \tau) \equiv 0$ and there is an interval of x for which $m'_{\widehat{v}}(x; \tau) > 0$.

3.2.1 Conditional Value Function for $\pi_1(\tau) \leq x_0 \leq \pi_2(\tau)$

We have shown that if $\pi_1(\tau) \leq x_0 \leq \pi_2(\tau)$, the consumer does not transfer any assets between the investment portfolio and the transactions account at time 0^+ . Therefore, the optimal value of X_{0^+} is X_0 and the optimal value of S_{0^+} is S_0 , so equation (30) implies that $\widehat{V}(X_0, S_0; \tau) \equiv F(X_0, S_0; \tau)$ for all (X_0, S_0) for which $\pi_1(\tau) \leq x_0 \equiv \frac{X_0}{S_0} \leq \pi_2(\tau)$. Hence, if

$\pi_1(\tau) \leq x_0 \leq \pi_2(\tau)$, then $\widehat{v}(x; \tau) \equiv f(x; \tau)$ and $m_{\widehat{v}}(x; \tau) \equiv m_f(x; \tau)$. Therefore, equations (35) and (39), together with the fact that $m_{\widehat{v}}(x; \tau) \equiv m_f(x; \tau)$ is strictly increasing in x , imply that

$$1 + \psi_b > m_{\widehat{v}}(x; \tau) > 1 - \psi_s, \quad \text{if } \pi_1(\tau) < x < \pi_2(\tau). \quad (43)$$

Figure 1 shows indifference curves of the conditional value function $\widehat{V}(X_0, S_0; \tau)$ and the restricted value function $F(X_0, S_0; \tau)$. The half-lines through the origin with slopes $\pi_1(\tau)$ and $\pi_2(\tau)$, respectively, separate the positive quadrant in Figure 1 into three regions: I, II, and III. The value of x at any point in the positive quadrant equals the slope of the line through that point and the origin. Thus, Region II, which is the cone bounded by the two half-lines with slopes $\pi_1(\tau)$ and $\pi_2(\tau)$, respectively, contains the combinations of X and S for which $\pi_1(\tau) \leq x_0 \leq \pi_2(\tau)$. In Region II, the indifference curves of the conditional value function $\widehat{V}(X_0, S_0; \tau)$ and the restricted value function $F(X_0, S_0; \tau)$ are identical to each other and are strictly convex.

3.2.2 Conditional Value Function for $x_0 < \pi_1(\tau)$

Suppose that the consumer observes the value of the investment portfolio at time 0 and that the state variables take on values X_0 and S_0 such that $x_0 \equiv \frac{X_0}{S_0} < \pi_1(\tau)$. Since the ratio of the transactions account to the investment portfolio is "too small", the consumer will want to transfer assets from the investment portfolio to the transactions account to increase x_{0+} to $\pi_1(\tau)$. That is, the consumer will want to transfer assets to the transactions account to achieve

$$X_{0+} = \pi_1(\tau) S_{0+}. \quad (44)$$

Substituting equation (44) into equation (6) and solving for S_{0+} yields

$$S_{0+} = \frac{X_0 + (1 - \psi_s) S_0}{1 - \psi_s + \pi_1(\tau)}, \quad (45)$$

which, along with the definition of $x_0 \equiv \frac{X_0}{S_0}$, implies

$$S_{0+} = \frac{1 - \psi_s + x_0}{1 - \psi_s + \pi_1(\tau)} S_0 < S_0. \quad (46)$$

Since the consumer can instantaneously change his overall holdings of assets from (X_0, S_0) to $(X_{0+}, S_{0+}) = (\pi_1(\tau) S_{0+}, S_{0+})$ at time 0^+ , we have

$$\widehat{V}(X_0, S_0; \tau) = \widehat{V}(\pi_1(\tau) S_{0+}, S_{0+}; \tau), \quad \text{if } x_0 \leq \pi_1(\tau), \quad (47)$$

which is the value-matching condition.

Since the conditional value function is homogeneous of degree $1 - \alpha$, we can rewrite equation (47) as

$$S_0^{1-\alpha} \widehat{V}(x_0, 1; \tau) = S_{0+}^{1-\alpha} \widehat{V}(\pi_1(\tau), 1; \tau), \quad \text{if } x_0 \leq \pi_1(\tau). \quad (48)$$

Recall from equation (29) that $\widehat{V}(x_0, 1; \tau) = \widehat{v}(x_0; \tau)$ and use equation (45) to rewrite equation (48) as

$$\widehat{v}(x_0; \tau) = \left[\frac{1 - \psi_s + x_0}{1 - \psi_s + \pi_1(\tau)} \right]^{1-\alpha} \widehat{v}(\pi_1(\tau); \tau), \quad \text{if } x_0 \leq \pi_1(\tau). \quad (49)$$

Note, in particular, that for $x_0 = 0$ we have

$$\widehat{v}(0; \tau) = \left[\frac{1 - \psi_s}{1 - \psi_s + \pi_1(\tau)} \right]^{1-\alpha} \widehat{v}(\pi_1(\tau); \tau). \quad (50)$$

Differentiate $\widehat{v}(x_0; \tau)$ in equation (49) with respect to x_0 to obtain

$$\widehat{v}'(x_0; \tau) = \frac{1 - \alpha}{1 - \psi_s + x_0} \widehat{v}(x_0; \tau), \quad \text{if } x_0 \leq \pi_1(\tau). \quad (51)$$

Substitute equation (51) into the definition of the marginal rate of substitution $m_{\widehat{v}}(x; \tau)$ in equation (41) to obtain

$$m_{\widehat{v}}(x_0; \tau) = 1 - \psi_s, \quad \text{if } x_0 \leq \pi_1(\tau). \quad (52)$$

Therefore, if $x_0 \leq \pi_1(\tau)$, the slope of the indifference curve of the value function $\widehat{V}(X_0, S_0; \tau)$ is constant and equal to $-(1 - \psi_s)$. In Figure 1, Region I, which is the set of points in the positive quadrant on and below the half-line with slope $\pi_1(\tau)$, is the set for which $x_0 \leq \pi_1(\tau)$. In Region I, the straight line with slope $-(1 - \psi_s)$ is an indifference curve of the conditional value function $\widehat{V}(X_0, S_0; \tau)$, and the curved dashed line is an indifference curve of the restricted value function $F(X_0, S_0; \tau)$. Note that these in-

difference curves meet at the half-line with slope $\pi_1(\tau)$, which is the boundary between Regions I and II, and the two indifference curves have equal slopes at this point. That is, $m_f(\pi_1(\tau); \tau) = m_{\widehat{v}}(\pi_1(\tau); \tau) = 1 - \psi_s$, which is the smooth-pasting condition.

When (X_0, S_0) is in Region I at an observation date, the optimal action is to sell some assets from the investment portfolio and transfer the proceeds to the transactions account instantly. Each dollar of assets sold from the investment portfolio increases the consumer's liquid assets by $1 - \psi_s$ dollars. That is, by decreasing S_{0+} by one dollar and moving one dollar to the left in Figure 1, the consumer can increase X_{0+} by $1 - \psi_s$ dollars and thus move upward by $1 - \psi_s$ dollars in Figure 1. Thus, the consumer can move leftward and upward along a line with slope $-(1 - \psi_s)$. In fact, the consumer will sell enough assets from the investment portfolio to increase the ratio x_t to π_1 and thus move to point A . Because the consumer can instantly reach point A from any point on the line with slope $-(1 - \psi_s)$ extending down and to the right from point A , the consumer's expected present value of lifetime utility at any such point is the same as at point A . Thus, all of these points lie on the same indifference curve of the conditional value function $\widehat{V}(X_0, S_0; \tau)$.

3.2.3 Conditional Value Function for $x_0 > \pi_2$

Suppose that the consumer observes the value of the investment portfolio at time 0 and that the state variables X_0 and S_0 are such that $x_0 \equiv \frac{X_0}{S_0} > \pi_2(\tau)$. Since the ratio of the transactions account to the investment portfolio is "too high", the consumer will want to use some of the liquid asset in the transactions account to purchase additional assets in the investment portfolio to reduce x_{0+} to $\pi_2(\tau)$. That is, at time 0^+ the consumer will transfer assets from the transactions account to the investment portfolio to achieve

$$X_{0+} = \pi_2(\tau) S_{0+}. \quad (53)$$

Substituting equation (53) into equation (7) and solving for S_{0+} yields

$$S_{0+} = \frac{X_0 + (1 + \psi_b) S_0}{1 + \psi_b + \pi_2(\tau)}, \quad (54)$$

which, along with $x_0 \equiv \frac{X_0}{S_0}$, implies

$$S_{0+} = \frac{1 + \psi_b + x_0}{1 + \psi_b + \pi_2(\tau)} S_0 > S_0. \quad (55)$$

Since the consumer can instantaneously re-allocate his holdings of assets from (X_0, S_0) to $(X_{0+}, S_{0+}) = (\pi_2(\tau) S_{0+}, S_{0+})$ at time 0^+ , we have

$$\widehat{V}(X_0, S_0; \tau) = \widehat{V}(\pi_2(\tau) S_{0+}, S_{0+}; \tau), \quad \text{if } x_0 \geq \pi_2(\tau), \quad (56)$$

which is the value-matching condition.

Since the value function is homogeneous of degree $1 - \alpha$, we can rewrite equation (56) as

$$S_0^{1-\alpha} \widehat{V}(x_0, 1; \tau) = S_{0+}^{1-\alpha} \widehat{V}(\pi_2(\tau), 1; \tau), \quad \text{if } x_0 \geq \pi_2(\tau). \quad (57)$$

Recall from equation (29) that $\widehat{V}(x_0, 1; \tau) = \widehat{v}(x_0; \tau)$ and use equation (54) to rewrite equation (57) as

$$\widehat{v}(x_0; \tau) = \left[\frac{1 + \psi_b + x_0}{1 + \psi_b + \pi_2(\tau)} \right]^{1-\alpha} \widehat{v}(\pi_2(\tau); \tau), \quad \text{if } x_0 \geq \pi_2(\tau). \quad (58)$$

Differentiate $\widehat{v}(x_0; \tau)$ in equation (58) with respect to x_0 to obtain

$$\widehat{v}'(x_0; \tau) = \frac{1 - \alpha}{1 + \psi_b + x_0} \widehat{v}(x_0; \tau), \quad \text{if } x_0 \geq \pi_2(\tau). \quad (59)$$

Substitute equation (59) into the definition of $m_{\widehat{v}}(x; \tau)$ in equation (41) to obtain

$$m_{\widehat{v}}(x; \tau) = 1 + \psi_b, \quad \text{if } x_0 \geq \pi_2(\tau). \quad (60)$$

Therefore, if $x_0 \geq \pi_2(\tau)$, the slope of the indifference curve of the conditional value function $\widehat{V}(X_0, S_0; \tau)$ is constant and equal to $-(1 + \psi_b)$. In Figure 1, Region III, which is the set of points in the positive quadrant on and above the half-line with slope $\pi_2(\tau)$, is the set for which $x_0 \geq \pi_2(\tau)$. In Region III, the straight line with slope $-(1 + \psi_b)$ is an indifference of the conditional value function $\widehat{V}(X_0, S_0; \tau)$, and the curved dashed line is an indifference curve of the restricted value function $F(X_0, S_0; \tau)$. These indifference curves meet, and have equal slopes, at the half-line with slope $\pi_2(\tau)$. Thus, $m_f(\pi_2(\tau); \tau) = m_{\widehat{v}}(\pi_2(\tau); \tau) = 1 + \psi_b$, which is the smooth-pasting condition.

When (X_0, S_0) is in Region III at an observation date, the optimal action is to use some of the liquid assets in the transactions account to increase the investment portfolio. The consumer must spend $1 + \psi_b$ of liquid assets from the transactions account to increase the investment portfolio by one dollar. That is, in order to increase S_{0+} by one dollar and move

one dollar to the right in Figure 1, the consumer can must decrease X_{0+} by $1 + \psi_b$ dollars and thus move downward by $1 + \psi_b$ dollars in Figure 1. Thus, the consumer can move rightward and downward along a line with slope $-(1 + \psi_b)$. In fact, the consumer will sell enough assets from the investment portfolio to decrease the ratio x_t to π_2 and thus move to point B . Because the consumer can instantly reach point B from any point on the line with slope $-(1 + \psi_b)$ extending up and to the left from point B , the consumer's expected present value of lifetime utility at any such point is the same as at point B . Thus, all of these points lie on the same indifference curve of the conditional value function $\widehat{V}(X_0, S_0; \tau)$.

3.3 Properties of the Value Function $V(X_t, S_t)$

Recall from equation (27) that the value function equals the conditional value function evaluated at the optimal value of τ . That is, $V(X_0, S_0) = \widehat{V}(X_0, S_0; \widehat{\tau}_0(X_0, S_0))$. The envelope theorem implies that $V_X(X_0, S_0) = \widehat{V}_X(X_0, S_0; \widehat{\tau}_0(X_0, S_0))$ and $V_S(X_0, S_0) = \widehat{V}_S(X_0, S_0; \widehat{\tau}_0(X_0, S_0))$. Thus, putting together the results from Regions I, II, and III, as summarized in equations (43), (52), and (60), we have for any observation date t_j

$$1 + \psi_b \geq \frac{V_S(X_{t_j}, S_{t_j})}{V_X(X_{t_j}, S_{t_j})} = m_v(x_\tau) \geq 1 - \psi_s \quad (61)$$

where

$$m_v(x) \equiv (1 - \alpha) \frac{v(x)}{v'(x)} - x \quad (62)$$

is the marginal rate of substitution between X_{t_j} and S_{t_j} for the value function $V(X_{t_j}, S_{t_j})$ and $v(x) = V(x, 1)$.

Since $V_X(X_{t_j}, S_\tau) > 0$, equation (61) implies that

$$(1 + \psi_b) V_X(X_{t_j}, S_{t_j}) \geq V_S(X_{t_j}, S_{t_j}) \geq (1 - \psi_s) V_X(X_{t_j}, S_{t_j}). \quad (63)$$

4 Optimal Share of Equity in the Investment Portfolio

The gross rate of return on the investment portfolio from time 0^+ to the next observation date τ , net of the observation cost at date τ , is $R(0, \tau)$, which is given by equation (4). The

excess rate of return on the investment portfolio, relative to a portfolio of riskless bonds, is

$$R(0, \tau) - (R^f)^\tau = \phi_0 \left[(1 - \theta) \frac{P_\tau}{P_0} - (R^f)^\tau \right]. \quad (64)$$

The consumer chooses ϕ_0 to maximize

$$E_0 \{V(X_\tau, S_\tau)\} = E_0 \left\{ V \left(X_\tau, \left[\phi_0 (1 - \theta) \frac{P_\tau}{P_0} + (1 - \phi_0) (R^f)^\tau \right] S_{0+} \right) \right\} \quad (65)$$

by differentiating the right hand side with respect to ϕ_0 and setting the derivative equal to zero to obtain

$$E_0 \left\{ V_S(X_\tau, S_\tau) \left[(1 - \theta) \frac{P_\tau}{P_0} - (R^f)^\tau \right] \right\} = 0. \quad (66)$$

Since ϕ_0 is constant over the interval of time from 0^+ to τ , equations (64) and (66) imply that

$$E_0 \{V_S(X_\tau, S_\tau) [R(0, \tau) - (R^f)^\tau]\} = 0, \quad (67)$$

which implies

$$E_0 \{V_S(X_\tau, S_\tau) R(0, \tau)\} = E_0 \{V_S(X_\tau, S_\tau)\} (R^f)^\tau. \quad (68)$$

5 Does the Consumer Consume All Liquid Assets Before the Next Observation Date?

In this section, we consider whether the consumer spends all of the liquid assets in the transactions account before the next observation date. As we will show in Section 7, this question is crucial for determining the long-run behavior of the optimal value of x_{t_j} .

Define

$$G(\tau) \equiv \frac{1 - \psi_s}{1 + \psi_b} (R^f)^\tau - (R^L)^\tau \quad (69)$$

as the net gain to the consumer from a round-trip transaction from the liquid asset in the transactions account to the riskless asset in the investment portfolio, and then back to the liquid asset in the transactions account on the next observation date. Specifically, the consumer reduces the transactions account by one dollar and uses this dollar to buy $\frac{1}{1 + \psi_b}$ dollars of the riskless bond in the investment portfolio at time 0^+ . This amount will grow to $\frac{1}{1 + \psi_b} (R^f)^\tau$ dollars at time τ and can be converted to $\frac{1 - \psi_s}{1 + \psi_b} (R^f)^\tau$ dollars of liquid assets in the transactions account at time τ . $G(\tau)$ is the excess return on this round-trip transaction

compared to leaving the dollar in the transactions account from time 0^+ to time τ and growing to $(R^L)^\tau$ dollars. This excess return can be negative, zero, or positive depending on the value of τ .

Define

$$\bar{\tau} \equiv \frac{1}{r_f - r_L} \ln \left(\frac{1 + \psi_b}{1 - \psi_s} \right) \quad (70)$$

and observe that $\bar{\tau}$ is positive provided that at least one of the transaction cost parameters, ψ_b and ψ_s , is positive. It follows directly from the definition of $G(\tau)$ in equation (69) that $G(\tau) \leq 0$ as $\tau \leq \bar{\tau}$. That is, for $0 \leq \tau < \bar{\tau}$, the higher return on the riskless bond in the investment portfolio would not accumulate for a long enough period of time to overcome the transactions costs associated with a round-trip from the transactions account to the investment portfolio and back. However, for $\tau > \bar{\tau}$, the higher return on the investment portfolio is earned for a long enough period of time to make $G(\tau) > 0$. Thus if $G(\tau) > 0$, the consumer will always benefit from transferring to the investment portfolio any liquid assets in the transactions account in excess of the amount needed to finance consumption until the next observation date. We will set up and analyze consumer's constrained maximization as a Lagrangian and will use this Lagrangian to analyze whether the consumer plans to hold any assets in the transactions account when the next observation date arrives. As we will show, the sign of $G(\tau)$ plays an important role in answering this question.

Suppose that time 0 is an observation date and that the next observation date τ is given. The consumer chooses C , X_{0^+} , S_{0^+} and ϕ_0 to maximize $U(C; \tau) + \beta^\tau E_0 \{V(X_\tau, S_\tau)\}$ where $X_\tau = (X_{0^+} - C)(R^L)^\tau$ and $S_\tau = R(0, \tau)S_{0^+}$. This maximization is subject to equations (6) and (7) and the constraint that $C \leq X_{0^+}$, which states that the consumer must pay for consumption using the transactions account. To set up a Lagrangian expression to allow us to determine if the constraint $C \leq X_{0^+}$ is strictly binding, it is helpful to introduce $\Delta S^{\text{buy}} \equiv \max[S_{0^+} - S_0, 0] \geq 0$ as the increase in the investment portfolio when the consumer uses some of the liquid assets in the transactions account to buy assets for the investment portfolio at time 0^+ , and $\Delta S^{\text{sell}} \equiv \min[S_{0^+} - S_0, 0] \leq 0$ as the (negative of the) decrease in the investment portfolio when the consumer sells assets from the investment portfolio and transfers the proceeds to the transactions account at time 0^+ . Taking account of the transactions costs associated with transferring assets between the investment portfolio and the transactions account, these definitions imply

$$X_{0^+} = X_0 - (1 - \psi_s) \Delta S^{\text{sell}} - (1 + \psi_b) \Delta S^{\text{buy}}. \quad (71)$$

The consumer's conditional value function, given an arbitrary value of τ , is

$$\begin{aligned} \widehat{V}(X_0, S_0; \tau) = & \max_{C, \Delta S^{\text{sell}}, \Delta S^{\text{buy}}, \phi_0} U(C; \tau) \\ & + \beta^\tau E_0 \{V((X_{0+} - C)(R^L)^\tau, R(0, \tau)S_{0+})\} \\ & + \lambda^{\text{buy}} \Delta S^{\text{buy}} - \lambda^{\text{sell}} \Delta S^{\text{sell}} + \eta [X_{0+} - C], \end{aligned} \quad (72)$$

where $\lambda^{\text{buy}} \geq 0$, $\lambda^{\text{sell}} \geq 0$, and $\eta \geq 0$ are Lagrange multipliers on the constraints $\Delta S^{\text{buy}} \geq 0$, $\Delta S^{\text{sell}} \leq 0$, and $X_{0+} - C \geq 0$, respectively. Substitute equations (71) and (117) into equation (72) to obtain

$$\begin{aligned} \widehat{V}(X_0, S_0; \tau) = & \max_{C, \Delta S^{\text{sell}}, \Delta S^{\text{buy}}, \phi_0} U(C; \tau) \\ & + \beta^\tau E_0 \left\{ V \left(\begin{array}{c} (X_0 - (1 - \psi_s) \Delta S^{\text{sell}} - (1 + \psi_b) \Delta S^{\text{buy}} - C) (R^L)^\tau, \\ R(0, \tau) (S_0 + \Delta S^{\text{sell}} + \Delta S^{\text{buy}}) \end{array} \right) \right\} \\ & + \lambda^{\text{buy}} \Delta S^{\text{buy}} - \lambda^{\text{sell}} \Delta S^{\text{sell}} \\ & + \eta [X_0 - (1 - \psi_s) \Delta S^{\text{sell}} - (1 + \psi_b) \Delta S^{\text{buy}} - C]. \end{aligned} \quad (73)$$

Differentiate the right hand side of equation (73) with respect to C , ΔS^{sell} , ΔS^{buy} , respectively, set the derivatives equal to zero, and use equation (68) to obtain

$$U'(C; \tau) = \beta^\tau E_0 \{V_X(X_\tau, S_\tau)\} (R^L)^\tau + \eta. \quad (74)$$

$$\begin{aligned} & \lambda^{\text{sell}} + (1 - \psi_s) \eta \\ = & -\beta^\tau E_0 \{V_X(X_\tau, S_\tau)\} (1 - \psi_s) (R^L)^\tau + \beta^\tau E_0 \{V_S(X_\tau, S_\tau)\} (R^f)^\tau. \end{aligned} \quad (75)$$

and

$$\begin{aligned} & -\lambda^{\text{buy}} + (1 + \psi_b) \eta \\ = & -(1 + \psi_b) \beta^\tau E_0 \{V_X(X_\tau, S_\tau)\} (R^L)^\tau + \beta^\tau E_0 \{V_S(X_\tau, S_\tau)\} (R^f)^\tau. \end{aligned} \quad (76)$$

Lemma 2 *If $\psi_b + \psi_s > 0$, then $\Delta S^{\text{sell}} = 0$ or $\Delta S^{\text{buy}} = 0$.*

Proof. Subtract equation (76) from equation (75) and use equation (74) to obtain $\lambda^{\text{sell}} + \lambda^{\text{buy}} = (\psi_b + \psi_s) U'(C; \tau)$. Thus, that as long as at least one of the transactions

cost parameters ψ_s and ψ_b is positive, the Lagrange multipliers λ^{sell} and λ^{buy} must sum to a positive number, so that at least one of the constraints $\Delta S^{\text{sell}} \leq 0$ or $\Delta S^{\text{buy}} \geq 0$ must bind.

■

Thus, in the presence of transactions costs, the consumer will not simultaneously buy and sell assets in the investment portfolio.

Lemma 3 *If $G(\tau) > 0$, then $C = X_{0+}$ and hence $X_\tau = 0$.*

Proof. Multiply and divide the first term on the right hand side of equation (76) by $1 - \psi_s$ and use $V_S(X_\tau, S_\tau) \geq (1 - \psi_s) V_X(X_\tau, S_\tau)$ from equation (63) to obtain $(1 + \psi_b) \eta \geq \beta^\tau E_0 \{V_S(X_\tau, S_\tau)\} \left[(R^f)^\tau - \frac{1 + \psi_b}{1 - \psi_s} (R^L)^\tau \right] + \lambda^{\text{buy}}$. Now use the definition of $G(\tau)$ in equation (69) to rewrite this equation as $\eta \geq \frac{1}{1 - \psi_s} \beta^\tau E_0 \{V_S(X_\tau, S_\tau)\} G(\tau) + \frac{\lambda^{\text{buy}}}{1 + \psi_b}$. Since $\psi_s < 1$, $\psi_b \geq 0$, $\beta^\tau E_0 \{V_S(X_\tau, S_\tau)\} > 0$, and $\lambda^{\text{buy}} \geq 0$, the assumption that $G(\tau) > 0$ implies $\eta > 0$, which implies $C = X_{0+}$, which implies $X_\tau = 0$. ■

The intuition underlying Lemma 3 is straightforward. As we have explained earlier, if $G(\tau) > 0$, the consumer can earn a positive riskless return from a round-trip transaction from the riskless liquid asset in the transactions account to the riskless asset in the investment portfolio, and then back to the liquid asset in the transactions account on the next observation date. Because of this opportunity for a riskless gain by transferring assets from the transactions account to the investment account, the consumer will transfer as many liquid assets as possible from the transactions account, while leaving enough liquid assets in the transactions account to finance consumption until the next observation date. Since consumer will hold only enough liquid assets in the transactions account to finance until the next observation date, the transactions account will be completely depleted on the next observation date. Thus, if the next observation date is time τ , then X_τ will be zero.

Next we will show that it is not necessary for $G(\tau) > 0$ in order for $C = X_{0+}$ and hence $X_\tau = 0$.

Lemma 4 *If $x_0 < \pi_1(\tau)$, then $C = X_{0+}$ and $X_\tau = 0$.*

Proof. If $x_0 < \pi_1(\tau)$, then the consumer sells assets from the investment portfolio and transfers the proceeds to the transactions account so that $x_{0+} = \pi_1(\tau)$. Since $\Delta S^{\text{sell}} > 0$, $\lambda^{\text{sell}} = 0$, so equation (75) implies $(1 - \psi_s) \eta = -\beta^\tau E_0 \{V_X(X_\tau, S_\tau)\} (1 - \psi_s) (R^L)^\tau + \beta^\tau E_0 \{V_S(X_\tau, S_\tau)\} (R^f)^\tau$. Use $V_S(X_\tau, S_\tau) \geq (1 - \psi_s) V_X(X_\tau, S_\tau)$ from equation 63) to obtain $(1 - \psi_s) \eta \geq \beta^\tau E_0 \{V_S(X_\tau, S_\tau)\} \times [(R^f)^\tau - (R^L)^\tau]$. Since $\beta^\tau E_0 \{V_S(X_\tau, S_\tau)\}, (R^f)^\tau >$

$(R^L)^\tau$, and $1 - \psi_s > 0$, we have $\eta > 0$. Since η is the Lagrange multiplier on the constraint $X_{0+} - C \geq 0$, $\eta > 0$ implies $C = X_{0+}$, which implies $X_\tau = 0$. ■

The economic intuition underlying Lemma 4 is straightforward. Since $x_0 < \pi_1(\tau)$, the consumer transfers assets from the investment portfolio to the liquid asset in the transactions account at time 0^+ . He will sell only enough of the investment portfolio to obtain enough of the liquid asset to finance consumption, C . He will not want to acquire additional liquid assets because he knows that he would arrive at time τ with a positive holding of the liquid asset. Instead of paying a transaction cost to acquire an additional dollar of the liquid asset and earn $(R^L)^\tau$ over the interval from time 0^+ to time τ , the consumer could leave the dollar in the investment portfolio and hold the dollar in riskless bonds earning $(R^f)^\tau$ over the interval from 0^+ to τ . Since $R^f > R^L$, the consumer will choose not to acquire the additional dollar of liquid asset at time 0^+ . That is, the consumer will acquire only enough of the liquid asset at time 0 to finance C , the present value of the consumption stream from 0^+ to τ . Note that this result holds regardless of the sign of $G(\tau)$.

Lemma 5 *Suppose that $x_0 > \pi_2(\tau)$. If $G(\tau) < 0$, then $C < X_{0+}$ and $X_\tau > 0$.*

Proof. We will use proof by contradiction. Suppose that $C = X_{0+}$, so $X_\tau = 0$, which implies $x_\tau = 0 < \pi_1(\tau)$. Therefore, equations (42) and (52) imply that $V_S(X_\tau, S_\tau) = (1 - \psi_s)V_X(X_\tau, S_\tau)$, which along with equation (76) and the definition of $G(\tau)$ in equation (69) implies $\eta = \frac{1}{1 - \psi_s}\beta^\tau E_0\{V_S(X_\tau, S_\tau)\}G(\tau) + \frac{\lambda^{\text{buy}}}{1 + \psi_b}$. Since $x_0 > \pi_2(\tau)$, the consumer uses some of the liquid asset in the transactions account to buy some assets for the investment portfolio. Therefore, $\Delta S^{\text{buy}} > 0$ which implies $\lambda^{\text{buy}} = 0$. Since, $1 + \psi_b > 0$, $1 - \psi_s > 0$, $\beta^\tau E_0\{V_S(X_\tau, S_\tau)\} > 0$, $\lambda^{\text{buy}} = 0$ and $G(\tau) < 0$, we have $\eta < 0$, which is a contradiction. ■

6 Dynamic Behavior of (X_t, S_t) : A Graphical Illustration

In this section we illustrate graphically the dynamic behavior implied by Lemmas 3 - 5. Suppose that time 0 is an observation date and that τ is the next observation date. We begin by considering various cases in which $G(\tau) < 0$.

First, consider the case in which $x_0 \equiv \frac{X_0}{S_0} > \pi_2$, so that (X_0, S_0) is in Region III. As shown in Figure 2, the consumer will instantaneously use some of the liquid assets in the transactions account to buy additional assets in the investment portfolio, reducing X_{0+} by $1 + \psi_b$ dollars

for every dollar that S_{0^+} is increased. That is, the consumer moves instantaneously from the point labeled "time 0" to the point labeled "time 0^+ ", which lies on the half-line with slope equal to π_2 . The liquid assets in the transactions account earn a known rate of return and since the consumer knows the entire consumption path from time 0^+ to time τ , the consumer at time 0 knows the value of the transactions account at time τ , X_τ . Since the value of X_τ is known at time 0, we know at time 0 that at time τ , (X_τ, S_τ) will lie somewhere along the horizontal dashed line in Figure 2. If the stock market performs very poorly between time 0 and time τ , so that S_τ is small, then (X_τ, S_τ) will be at a point such as that labeled "III" in Figure 2. That is, if the stock market performs poorly, (X_τ, S_τ) will be in Region III. Alternatively, if the stock market performs moderately well between time 0 and time τ , then (X_τ, S_τ) will be represented by a point such as that labeled "II" in Figure 2, which is in Region II. Finally, if the stock market performs very well between time 0 and time τ , then (X_τ, S_τ) will be represented by a point such as that labeled "I" in Figure 2, which is in Region I. To summarize, if $G(\tau) < 0$, then starting in Region III at time 0, the combination (X_τ, S_τ) can be in Region I, II, or III at the next observation date, time τ . If (X_τ, S_τ) is in Region III, then Figure 2 applies again. Figures 3 and 4 illustrate what happens if the consumer finds that (X_t, S_t) is in Regions II or I, respectively, on an observation date.

Continue to assume that $G(\tau) < 0$, so that $X_\tau > 0$, and now suppose that (X_0, S_0) is in Region II. In Region II, $\pi_1 < x_0 < \pi_2$ so the consumer does not transfer any assets between the transactions account and the investment portfolio at time 0. Thus, the point labeled "time 0 and time 0^+ " in Figure 3 shows the values of the transactions account and the investment portfolio at time 0 and time 0^+ . As in Figure 2, on an observation date. As we will see, Figure 4 is strikingly different from Figure 2, since the value of X_τ is known at time 0, we know at time 0 that at time τ , (X_τ, S_τ) will lie somewhere along the horizontal dashed line in Figure 3. If the stock market performs very poorly between time 0 and time τ , so that S_τ is small, then (X_τ, S_τ) will be at a point such as that labeled "III" in Figure 3, which is in Region III. If the stock market performs moderately well between time 0 and time τ , then (X_τ, S_τ) will be represented by a point such as that labeled "II" in Figure 3, which is in Region II. Finally, if the stock market performs very well between time 0 and time τ , then (X_τ, S_τ) will be represented by a point such as that labeled "I" in Figure 3, which is in Region I. Thus, just as in Figure 2 where consumer starts in Region III at time 0, Figure 3 illustrates that if $G(\tau) < 0$, then starting in Region III at time 0, the combination (X_τ, S_τ) can be in Region I, II, or III at the next observation date, time τ . If (X_τ, S_τ) is in Region III, then Figure 2 applies again and if (X_τ, S_τ) is in Region II, then Figure 3 applies again.

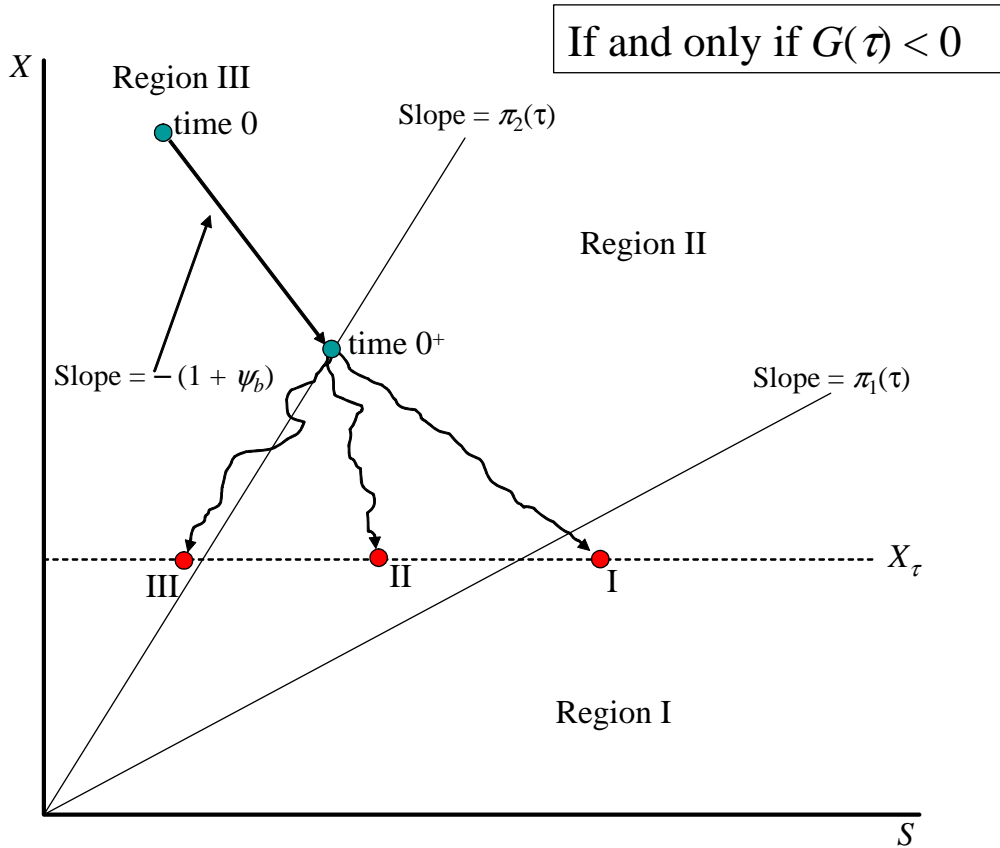


FIGURE 2:

Figure 4 illustrates what happens when the consumer finds that (X_t, S_t) is in Regions I on an observation date. As we will see, Figure 4 is strikingly different from Figures 2 and 3.

Now suppose that (X_0, S_0) is in Region I, where $x_0 < \pi_1$. Since $x_0 < \pi_1$, the consumer instantaneously sells some the assets in the investment portfolio, and transfer the proceeds, net of transactions costs, to the transactions portfolio. For each dollar of assets in the investment portfolio that the consumer sells, the transactions account will increase by $1 - \psi_s$ dollar. Thus, the consumer moves instantaneously from the point labeled "time 0" to the point labeled "time 0+", which lies along the half-line with slope equal to π_1 in Figure 4. Lemma 4 implies that since $x_0 < \pi_1$, the value of the transactions account at the next observation date, X_τ , will be zero. Thus, (X_τ, S_τ) will lie somewhere along the horizontal axis, regardless of the performance of the stock market.⁹ Figure 4 shows two possible

⁹This statement assumes that not more than 100% of the investment portfolio is held in risky equity. In Region I, the consumer plans to arrive at the next observation date with a zero balance in the transactions

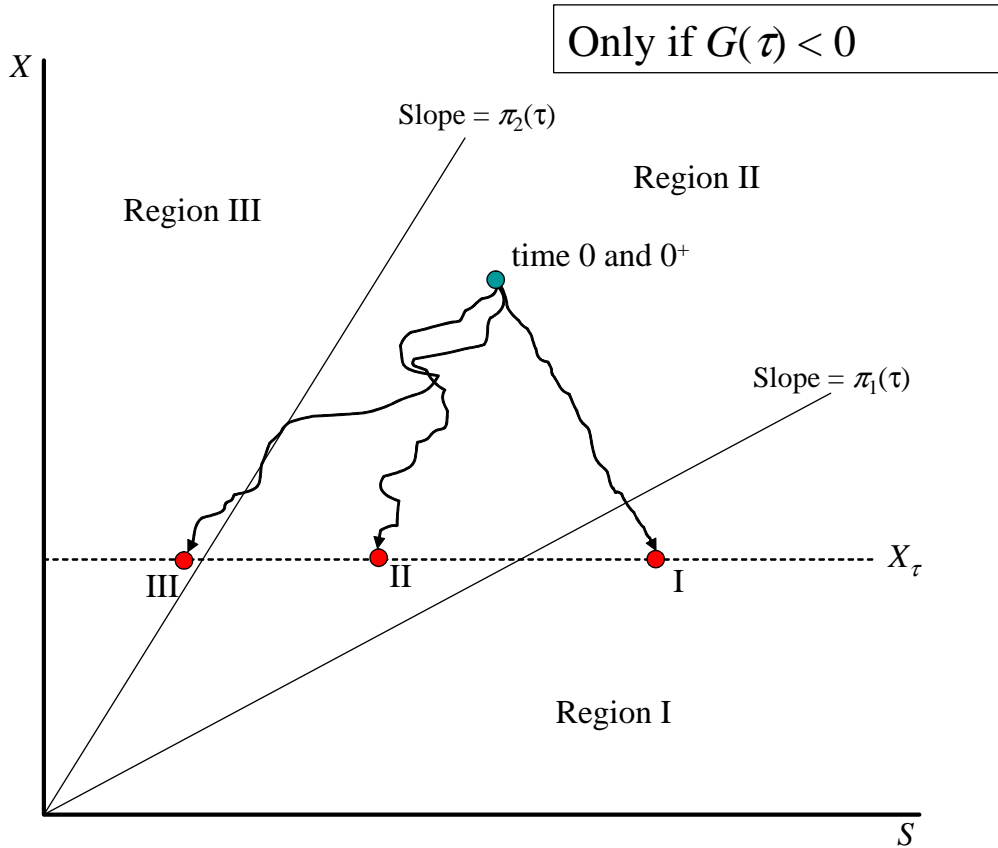


FIGURE 3:

points along the horizontal axis—both labeled "time τ "—which correspond to weak or strong performance of the stock market. Regardless of whether S_τ is relatively low or high, the consumer will at time τ^+ instantaneously sell assets from the investment portfolio and return to the half-line with slope $-(1 - \psi_s)$ in Figure 4. Then the process repeats itself over, and over again. Once the consumer is in Region I on observation date, the consumer will be along the horizontal axis, with a zero balance in the transactions account, on all subsequent observation dates.

So far, we have assumed that $G(\tau) < 0$. If $G(\tau) > 0$, then Lemma 3 implies that on the next observation date, the transactions account will have a zero balance and the consumer will be somewhere along the horizontal axis. From that time forward, Figure 4 applies and account, so the consumer's entire wealth will be in the investment portfolio. Since $V_S(0, S)$ approaches infinity as S approaches 0 from above, the consumer will make sure to devote a non-negative portion of the investment portfolio to riskless bonds to make sure that S does not fall to 0 or below.

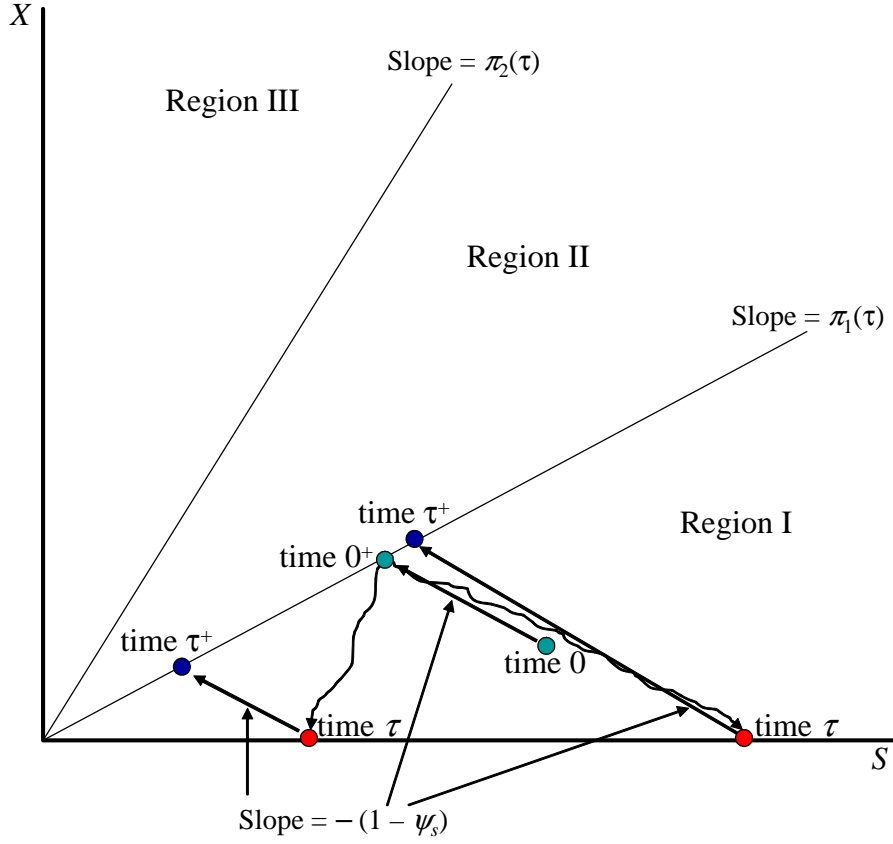


FIGURE 4:

the consumer remains in Region I.

7 Long-Run Behavior

This section formalizes the dynamic behavior of the transactions account and the investment portfolio, which were illustrated graphically in Section 6.

Proposition 1 (1) If $x_{t_j} \leq \pi_1$, then $x_{t_{j+1}} = 0$. (2) If $\pi_1 \leq x_{t_j} \leq \pi_2$, and if $G(\tau) > 0$, then $x_{t_{j+1}} = 0$. (3) If $x_{t_j} > \pi_2$, then $x_{t_{j+1}} = 0$ if $G(\tau) > 0$, and $x_{t_{j+1}} > 0$ if $G(\tau) < 0$.

Proof. Lemma 4 implies statement (1). Lemma 3 implies statement (2). Lemmas 3 and 5 imply statement (3). ■

Corollary 1 If $x_{t_j} \leq \pi_1$, then $x_{t_{j+i}} = 0$ for all $i = 1, 2, 3, \dots$

Corollary 2 *If $G(\tau) > 0$ at observation time t_j , then $x_{t_{j+i}} = 0$ for all $i = 1, 2, 3, \dots$*

Proposition 2 *If $x_{t_j} \geq \pi_2$, and if $G(\tau_j) < 0$ at observation time t_j , then $\Pr \{x_{t_{j+1}} \leq \pi_1\} < 1$.*

Proof. First, apply the envelope theorem to equation (17) to obtain: $V_S(X_{t_j}, S_{t_j}) = e^{-\rho\tau_j} E_{t_j} \{V_S(X_{t_{j+1}}, S_{t_{j+1}})\} (R^f)^\tau$ and $V_X(X_{t_j}, S_{t_j}) = e^{-\rho\tau_j} E_{t_j} \{V_X(X_{t_{j+1}}, S_{t_{j+1}})\} (R^L)^\tau$ and then use equation (61) to obtain $m_v(x_{t_j}) = \frac{V_S(X_{t_j}, S_{t_j})}{V_X(X_{t_j}, S_{t_j})} = \frac{E_{t_j} \{V_S(X_{t_{j+1}}, S_{t_{j+1}})\} (R^f)^\tau}{E_{t_j} \{V_X(X_{t_{j+1}}, S_{t_{j+1}})\} (R^L)^\tau}$. Now rewrite this equation using the fact that $V_X(X_{t_{j+1}}, S_{t_{j+1}}) = \frac{1}{m_v(x_{t_{j+1}})} V_S(X_{t_{j+1}}, S_{t_{j+1}})$ to obtain $m_v(x_{t_j}) E_{t_j} \left\{ \frac{1}{m_v(x_{t_{j+1}})} \frac{V_S(X_{t_{j+1}}, S_{t_{j+1}})}{E_{t_j} \{V_S(X_{t_{j+1}}, S_{t_{j+1}})\}} \right\} = \left(\frac{R^f}{R^L} \right)^\tau$. Since $x_{t_j} \geq \pi_2$, we know that $m_v(x_{t_j}) = 1 + \psi_b$. Multiplying both sides of (??) by $1 - \psi_s$, dividing both sides by $m_v(x_{t_j}) = 1 + \psi_b$ and rearranging yields $E_{t_j} \left\{ \frac{1 - \psi_s}{m_v(x_{t_{j+1}})} \frac{V_S(X_{t_{j+1}}, S_{t_{j+1}})}{E_{t_j} \{V_S(X_{t_{j+1}}, S_{t_{j+1}})\}} \right\} = \frac{1 - \psi_s}{1 + \psi_b} \left(\frac{R^f}{R^L} \right)^\tau < 1$. From this point on, the proof proceeds by contradiction. Suppose—counterfactually—that $\Pr \{x_{t_{j+1}} \leq \pi_1\} = 1$. In this case $m_v(x_{t_{j+1}}) = 1 - \psi_s$ so $E_{t_j} \left\{ \frac{1 - \psi_s}{m_v(x_{t_{j+1}})} \frac{V_S(X_{t_{j+1}}, S_{t_{j+1}})}{E_{t_j} \{V_S(X_{t_{j+1}}, S_{t_{j+1}})\}} \right\} = 1$, which contradicts the previous statement that $E_{t_j} \left\{ \frac{1 - \psi_s}{m_v(x_{t_{j+1}})} \frac{V_S(X_{t_{j+1}}, S_{t_{j+1}})}{E_{t_j} \{V_S(X_{t_{j+1}}, S_{t_{j+1}})\}} \right\} < 1$. Hence, it must be the case that $\Pr \{x_{t_{j+1}} \leq \pi_1\} < 1$, as asserted. ■

Recall that the consumer observes the value of the investment portfolio at dates t_j , $j = 0, 1, 2, \dots$. Let $x_{t_j^+}$ be the value of x immediately after the consumer observes the value of the investment portfolio at date t_j and optimally transfers assets between the transactions account and the investment portfolio. The sequence $x_{t_j^+}$, $j = 0, 1, 2, \dots$, is a stochastic process that is confined to a closed interval $[\pi_1, \pi_2]$.

Proposition 3 *If $x_{t_n^+} = \pi_1$, then $x_{t_j^+} = \pi_1$ for all $j > n$.*

Proof. If $x_{t_n^+} = \pi_1$, then $x_{t_{n+1}} = 0$, which implies $x_{t_{n+1}^+} = \pi_1$. ■

Proposition 3 states that π_1 is an absorbing value for the stochastic process $x_{t_j^+}$, which is the value of x_t immediately after observing the value of the investment portfolio and optimally transferring assets between the investment portfolio and the transactions account. The proof is straightforward: If the value of x_{t_j} is ever observed to be less than or equal to π_1 , the consumer immediately sells some of the assets in the investment portfolio and increases the transactions account so that $x_{t_j^+} = \pi_1$. When $x_{t_j^+} = \pi_1$, the consumer consumes the entire transactions account over the interval of time until t_{j+1} and so arrives at time t_{j+1} with a zero balance in the transactions account. Thus, $x_{t_{j+1}} = 0 < \pi_1$, so at time t_{j+1}

the consumer sells assets from the investment portfolio to make $x_{t_{j+1}^+} = \pi_1$, and the process repeats itself.

Lemma 6 *The distribution of $S_{t_{j+1}}$ conditional on ϕ_j and $S_{t_j^+}$ is a translated lognormal distribution with support $\left((1 - \phi_j) (R^f)^{\tau_j} S_{t_j^+}, \infty \right)$.*

Proof. Equation (3) implies $S_{t_{j+1}} = (1 - \theta) \phi_j \frac{P_{t_{j+1}}}{P_{t_j}} S_{t_j^+} + (1 - \phi_j) (R^f)^{\tau_j} S_{t_j^+}$. Since $\mu > r_f$, the optimal value of ϕ_j is positive. Also, $1 - \theta > 0$ and $S_{t_j^+} > 0$. $\frac{P_{t_{j+1}}}{P_{t_j}} > 0$ is conditionally lognormal with support $(0, \infty)$, so $(1 - \theta) \phi_j \frac{P_{t_{j+1}}}{P_{t_j}} S_{t_j^+}$ is also conditionally lognormal with support $(0, \infty)$, so $(1 - \theta) \phi_j \frac{P_{t_{j+1}}}{P_{t_j}} S_{t_j^+}$. Conditional on ϕ_j and $S_{t_j^+}$, $(1 - \phi_j) (R^f)^{\tau_j} S_{t_j^+}$ is a constant. ■

Proposition 4 *Let t_n be the first time that $x_{t_n^+} = \pi_1$. Then $\Pr \{t_n < \infty\} = 1$ and $E(t_n) < \infty$.*

Proof. Lemma 6 implies that $\delta(x) \equiv \Pr \left\{ x_{t_{j+1}} \leq \pi_1 | x_{t_j^+} = x \right\} > 0$ for any x in $[\pi_1, \pi_2]$. This probability satisfies $\delta(x) > 0$ and is continuous in x for $x \in [\pi_1, \pi_2]$.¹⁰ Since $[\pi_1, \pi_2]$ is closed and bounded (hence compact), there is a $\delta^* > 0$ such that $\delta(x) \geq \delta^*$ for $\pi_1 \leq x \leq \pi_2$. Suppose that $x_{t_0^+} = x > \pi_1$. We will show that:

$$\Pr(t_n > t_k) \leq (1 - \delta^*)^k \text{ for } k = 1, 2, 3, \dots \quad (77)$$

To show (77) it is easiest to employ an induction argument. First, note that (77) holds for $k = 1$, since $\Pr \{t_n > t_1\} = \Pr \left\{ x_{t_1} > \pi_1 | x_{t_0^+} = x \right\} < 1 - \delta^*$. Second, as we show next, if (77) holds for k , then it must also hold for $k + 1$. To see this, observe that

$$\Pr(t_n > t_{k+1}) = \Pr(t_n > t_{k+1}; t_n > t_k), \quad (78)$$

¹⁰We give only a sketch of the proof of these arguments. First note that the theorem of the maximum (See e.g. Stokey and Lucas (1989)) along with strict concavity of $E_{t_k} \{V(X_{t_k+\tau}, S_{t_k+\tau})\}$ implies that the portfolio share ϕ_0^* is continuous in x . This implies that $\delta(x)$ is continuous in x . Second, it is straightforward to show that $\phi_0 \neq 0$. To see this, suppose otherwise. Then

$$E_0 \left\{ V_S(X_\tau, S_\tau) \left[(1 - \theta) \frac{P_\tau}{P_0} - (R^f)^\tau \right] \right\} = V_S(X_\tau, S_\tau) E \left[(1 - \theta) \frac{P_\tau}{P_0} - (R^f)^\tau \right]$$

Since S_τ is deterministic. Assuming that $E \left[(1 - \theta) \frac{P_\tau}{P_0} - (R^f)^\tau \right] > 0$, it follows that $\phi = 0$ cannot be optimal.

because t_n can only be larger than t_{k+1} if it is also larger than t_k . Using (78) we have:

$$\begin{aligned}
\Pr(t_n > t_{k+1}) &= \frac{\Pr(t_n > t_{k+1}; t_n > t_k)}{\Pr(t_n > t_k)} \Pr(t_n > t_k) \\
&= \Pr(x_{t_{k+1}} > \pi_1 | x_{t_k} > \pi_1) \Pr(t_n > t_k) \\
&\leq (1 - \delta^*) (1 - \delta^*)^k = (1 - \delta^*)^{k+1}
\end{aligned} \tag{79}$$

where the second line follows from Bayes Rule, while the third line follows from the construction of δ^* and the inductive assumption (77). In light of (79), equation (77) follows by induction. Letting $k \rightarrow \infty$ in (77) we obtain $\Pr\{t_n < \infty\} = 1$. Finally

$$\begin{aligned}
E(t_n) &= \sum_{k=1}^{\infty} k \Pr(t_n = t_k) \\
&= 1 + \sum_{k=1}^{\infty} \Pr(t_n > t_k) \\
&\leq 1 + \sum_{k=1}^{\infty} (1 - \delta^*)^k = \frac{1}{\delta^*} < \infty
\end{aligned}$$

The second line follows by applying summation by parts¹¹ and the last line by using (77). ■

8 Behavior When the Transactions Balance Equals Zero on an Observation Date

We have shown that in the long run, the transactions balance will be zero on all observation dates, and the consumer will sell assets from the investment portfolio to increase the ratio of the transactions account balance to the value of the investment portfolio to π_1 . In this section, we focus on this long-run situation and derive an expression for the ratio π_1 and characterize the optimal interval of time between successive observations of the stock market.

It will be convenient to define

$$J(\tau) \equiv \beta^\tau E_0 \{ [R(t_j, t_j + \tau)]^{1-\alpha} \}, \tag{80}$$

¹¹ $\sum_{k=1}^{\infty} k \Pr(t_n = t_k) = \sum_{k=1}^{\infty} \sum_{j=1}^k \Pr(t_n = t_k) = \sum_{j=1}^{\infty} \sum_{k=j}^{\infty} \Pr(t_n = t_k) = \sum_{j=1}^{\infty} (\Pr(t_n = t_j) + \sum_{k=j+1}^{\infty} \Pr(t_n = t_k)) = \sum_{j=1}^{\infty} (\Pr(t_n = t_j) + \Pr(t_n > t_j)) = 1 + \sum_{j=1}^{\infty} \Pr(t_n > t_j)$.

where t_j is an observation date and τ is the length of time until the next observation date. The definition of $J(\tau)$ holds only in Region I. Recall that once the consumer reaches Region I on an observation date, the consumer will be in Region I at all future observation dates, and will choose the same allocation of the investment portfolio and the same interval of time until the next observation at all future observation dates. Thus, the distribution of $R(t_j, t_j + \tau)$, and hence $J(\tau)$, is invariant to t_j after the consumer has reached Region I. We assume that $J(\tau) < 1$, so that the value function is finite.

We first use this definition to obtain an expression for π_1 as a function of τ .

Lemma 7 $\pi_1(\tau) = \left(\frac{1-\psi_s}{v(0)}\right)^{\frac{1}{\alpha}} h(\tau) [J(\tau)]^{-\frac{1}{\alpha}}$.

Proof. See Appendix. ■

Lemma 7 is not a complete solution for $\pi_1(\tau)$ because it depends on $v(0)$. Nevertheless, it will prove helpful in solving for τ^* , the optimal value of τ , when the transactions account balance is zero on an observation date, and for the value of $\pi_1(\tau^*)$. As a step toward calculating τ^* , the following lemma presents an expression for the value conditional value function evaluated at $x = 0$.

Lemma 8 $\hat{v}(0; \tau) = \left[1 + \frac{\pi_1(\tau)}{1-\psi_s}\right]^{\alpha} J(\tau) v(0)$.

Proof. See Appendix. ■

Lemma 8 immediately allows us to calculate the value of the $\pi_1(\tau^*)$.

Proposition 5 $\pi_1(\tau^*) = (1 - \psi_s) \left([J(\tau^*)]^{-\frac{1}{\alpha}} - 1\right)$.

Proof. Recall that $\tau^* = \arg \max_{\tau} \hat{v}(0; \tau)$ so $v(0) = \hat{v}(0; \tau^*) = \left[1 + \frac{\pi_1(\tau^*)}{1-\psi_s}\right]^{\alpha} J(\tau^*) v(0)$. Therefore, $1 = \left[1 + \frac{\pi_1(\tau^*)}{1-\psi_s}\right]^{\alpha} J(\tau^*)$, so $1 + \frac{\pi_1(\tau^*)}{1-\psi_s} = [J(\tau^*)]^{-\frac{1}{\alpha}}$, which implies $\pi_1(\tau^*) = (1 - \psi_s) \left([J(\tau^*)]^{-\frac{1}{\alpha}} - 1\right)$. ■

Proposition 5 expresses $\pi_1(\tau^*)$ as in terms of $J(\tau^*)$, but we still need to determine τ^* . The following proposition provides a nonlinear equation that τ^* must satisfy.

Proposition 6 *If the transactions balance is zero on an observation date, the optimal time until the next observation, τ^* , satisfies $\frac{h'(\tau^*)}{h(\tau^*)} = \frac{1}{\alpha} \frac{1}{1-[J(\tau^*)]^{-\frac{1}{\alpha}}} \frac{J'(\tau^*)}{J(\tau^*)}$.*

Proof. See Appendix. ■

τ	AEP1	AEP2
Baseline	0.696	0.399
$\theta = 0.001$	2.223	1.267
$\rho = 0.02$	0.662	0.383
$\alpha = 2$	0.587	0.509
$r^L = 0$	0.557	0.328
$r = 0.03$	0.541	0.279
$\mu = 0.07$	0.584	0.381
$\sigma = 0.20$	0.796	0.341

Corollary 3 *If the transactions balance is zero on an observation date, the optimal time until the next observation, τ^* , is invariant to the transactions cost parameters ψ_s and ψ_b .*

Corollary 4 *Suppose that the transactions balance is zero on observation date t_j . Let $\{\tilde{c}_t\}_{t=t_j^+}^\infty$ be the path of optimal future consumption if $\psi_s = \psi_b = 0$. Then for arbitrary ψ_s and ψ_b the path of optimal future consumption is $\{(1 - \psi_s)\tilde{c}_t\}_{t=t_j^+}^\infty$.*

Corollary 4 shows that once the consumer has reached a zero transactions balance on an observation date, the transactions cost parameter ψ_s can be viewed as pure consumption tax that does not affect the timing of observations nor the amount of the investment portfolio that is sold on each observation date. However, an increase in ψ_s reduces amount by which the transactions account balance increases as a result of any given sale of assets from the investment portfolio.

8.1 Quadratic Approximation

In order to see how the optimal value of τ depends on the various parameters of the consumer's problem, we will approximate the nonlinear equation describing τ^* in Proposition 6. As we will show, this equation is locally quadratic in τ around $\tau = 0$. Therefore, instead of a linear approximation, we will need a quadratic approximation.

Before proceeding to the quadratic approximation, we will define

$$\chi \equiv (1 - \theta)^{\frac{1-\alpha}{\alpha}}. \quad (81)$$

This transformation of the observation cost parameter will prove particular convenient when we compare the results to those in AEP. Notice that when the observation cost, θ , is zero, $\chi = 1$. For $0 < \theta < 1$, $\chi < 1$ if $\alpha < 1$ and $\chi > 1$ if $\alpha > 1$.

Define the function $M(\tau, \chi)$ as

$$M(\tau, \chi) \equiv \left[1 - \frac{1}{\alpha} h(\tau) [h'(\tau)]^{-1} [J(\tau, \chi)]^{-1} J_\tau(\tau, \chi) \right] [J(\tau, \chi)]^{\frac{1}{\alpha}}, \quad (82)$$

where the function $J(\tau)$ in equation (80) can be rewritten using the portfolio rate of return in equation (4) and the definition of χ in equation (81) as

$$J(\tau, \chi) \equiv \beta^\tau E_0 \left\{ \left[\chi^{\frac{\alpha}{1-\alpha}} \phi_0 \frac{P_\tau}{P_0} + (1 - \phi_0) (R^f)^\tau \right]^{1-\alpha} \right\}. \quad (83)$$

Observe that when $M(\tau, \chi) = 1$ the nonlinear equation in Proposition 6 for the optimal value of τ is satisfied. Our strategy is to approximate the function $M(\tau, \chi)$ around $(\tau, \chi) = (0, 1)$. The following functions evaluated at $(\tau, \chi) = (0, 1)$ will be helpful:

$$h(0) = 0 \quad (84)$$

$$h'(0) = 1 \quad (85)$$

$$h''(0) = -\omega \quad (86)$$

$$J(0, 1) = 1 \quad (87)$$

Equations (84) and (87) imply that

$$M(0, 1) = 1. \quad (88)$$

Differentiate equation (82) with respect to τ to obtain

$$M_\tau(\tau, \chi) = -\frac{1}{\alpha} \frac{h(\tau)}{h'(\tau)} \left[\left(\frac{1}{\alpha} - 1 \right) \frac{J_\tau(\tau, \chi)}{J(\tau, \chi)} - \frac{h''(\tau)}{h'(\tau)} + \frac{J_{\tau\tau}(\tau, \chi)}{J_\tau(\tau, \chi)} \right] [J(\tau, \chi)]^{\frac{1}{\alpha}} \frac{J_\tau(\tau, \chi)}{J(\tau, \chi)}. \quad (89)$$

Evaluate equation (89) at $\tau = 0$, and use the fact that $h(0) = 0$, to obtain

$$M_\tau(0, \chi) = 0. \quad (90)$$

Differentiate equation (89) with respect to τ , and evaluate the expression at $\tau = 0$ using

the facts that $h(0) = 0$, $h'(0) = 1$, and $h''(0) = -\omega$ to obtain

$$M_{\tau\tau}(0, \chi) = -\frac{1}{\alpha} \left[\left(\frac{1}{\alpha} - 1 \right) \frac{J_\tau(0, \chi)}{J(0, \chi)} + \omega + \frac{J_{\tau\tau}(0, \chi)}{J_\tau(0, \chi)} \right] [J(0, \chi)]^{\frac{1}{\alpha}} \frac{J_\tau(0, \chi)}{J(0, \chi)} \quad (91)$$

Now evaluate equation (91) at $\chi = 1$ and use the fact that $J(0, 1) = 1$ to obtain

$$M_{\tau\tau}(0, 1) = -\frac{1}{\alpha} \left[\omega J_\tau(0, 1) + \left(\frac{1}{\alpha} - 1 \right) [J_\tau(0, 1)]^2 + J_{\tau\tau}(0, 1) \right] \quad (92)$$

Now differentiate equation (82) with respect to χ to obtain

$$M_\chi(\tau, \chi) = \frac{1}{\alpha} \left(\frac{h(\tau)}{h'(\tau)} \left[\left(1 - \frac{1}{\alpha} \right) \frac{J_\chi(\tau, \chi)}{J(\tau, \chi)} \frac{J_\tau(\tau, \chi)}{J(\tau, \chi)} - \frac{J_{\tau\chi}(\tau, \chi)}{J(\tau, \chi)} \right] + \frac{J_\chi(\tau, \chi)}{J(\tau, \chi)} \right) [J(\tau, \chi)]^{\frac{1}{\alpha}} \quad (93)$$

Evaluate equation (93) at $\tau = 0$ using the fact that $h(0) = 0$ to obtain

$$M_\chi(0, \chi) = \frac{1}{\alpha} J_\chi(\tau, \chi) [J(\tau, \chi)]^{\frac{1}{\alpha}-1} \quad (94)$$

Evaluate equation (94) at $\chi = 1$ using the fact that $J(0, 1) = 1$ to obtain

$$M_\chi(0, \chi) = \frac{1}{\alpha} J_\chi(0, 1) \quad (95)$$

Differentiate equation (93) with respect to τ and evaluate the derivative at $\tau = 0$ to obtain

$$M_{\chi\tau}(0, \chi) = 0. \quad (96)$$

Differentiate equation (93) with respect to χ and evaluate the derivative at $\tau = 0$ to obtain

$$M_{\chi\chi}(0, \chi) = \frac{1}{\alpha} \left[\frac{J_{\chi\chi}(0, \chi)}{J(0, \chi)} + \left(\frac{1}{\alpha} - 1 \right) \left(\frac{J_\chi(0, \chi)}{J(0, \chi)} \right)^2 \right] [J(0, \chi)]^{\frac{1}{\alpha}} \quad (97)$$

Evaluate equation (97) at $\chi = 1$ using the fact that $J(0, \chi) = 1$ to obtain

$$M_{\chi\chi}(0, 1) = \frac{1}{\alpha} \left[J_{\chi\chi}(0, 1) + \left(\frac{1}{\alpha} - 1 \right) [J_\chi(0, 1)]^2 \right] \quad (98)$$

Therefore, the second-order Taylor expansion of $M(\tau, \chi)$ around $(\tau, \chi) = (0, 1)$, denoted

$\widehat{M}(\tau, \chi)$, is

$$\widehat{M}(\tau) \equiv 1 + M_\chi(0, 1)(\chi - 1) + \frac{1}{2}(M_{\tau\tau}(0, 1)\tau^2 + M_{\chi\chi}(0, 1)(\chi - 1)^2) \quad (99)$$

Define $\widehat{\tau}$ as $\widehat{M}(\widehat{\tau}) = 1$ so

$$\widehat{\tau} = \sqrt{-\frac{2M_\chi(0, 1)(\chi - 1) + M_{\chi\chi}(0, 1)(\chi - 1)^2}{M_{\tau\tau}(0, 1)}} \quad (100)$$

9 Appendix

Proof of Lemma 1: We will first prove that if $V(X, S)$ is concave, then $F(X_0, S_0; \tau)$ is strictly concave for any τ . To prove this, let C^* maximize the right hand side of equation (20) for $X_0 = X^*$ and $S_0 = S^*$, which implies $X_\tau^* = e^{rL\tau}(X^* - C^*)$ and $S_\tau^* = R(0, \tau)S^*$. Let C^{**} maximize the right hand side of equation (20) for $X_0 = X^{**}$ and $S_0 = S^{**}$, which implies $X_\tau^{**} = e^{rL\tau}(X^{**} - C^{**})$ and $S_\tau^{**} = R(0, \tau)S^{**}$. Now consider $X_0 = X^{***} \equiv aX^* + (1-a)X^{**}$ and $S_0 = S^{***} \equiv aS^* + (1-a)S^{**}$ for $0 < a < 1$, and define $C^{***} = aC^* + (1-a)C^{**}$, which implies $X_\tau^{***} = aX_\tau^* + (1-a)X_\tau^{**}$ and $S_\tau^{***} = aS_\tau^* + (1-a)S_\tau^{**}$. Therefore, $F(X^{***}, S^{***}; \tau) \geq U(C^{***}) + e^{-\rho\tau} E\{V(X_\tau^{***}, S_\tau^{***})\}$. Now use the facts that $U(C^{***}) > aU(C^*) + (1-a)U(C^{**})$ by the strict concavity of $U(C)$ and $V(X_\tau^{***}, S_\tau^{***}) \geq aV(X_\tau^*, S_\tau^*) + (1-a)V(X_\tau^{**}, S_\tau^{**})$ by the weak concavity of $V(X, S)$ to obtain $F(X^{***}, S^{***}; \tau) > aU(C^*) + (1-a)U(C^{**}) + e^{-\rho\tau} [aV(X_\tau^*, S_\tau^*) + (1-a)V(X_\tau^{**}, S_\tau^{**})] = aF(X^*, S^*; \tau) + (1-a)F(X^{**}, S^{**}; \tau)$. Therefore, $F(X_0, S_0; \tau)$ is strictly concave in X_0 and S_0 . [This not the end of the proof]

To prove the concavity of V , let $\vartheta(X_t, S_t)$ be an arbitrary increasing, concave function that is homogeneous of degree $1 - \alpha$ in X_t and S_t so that it can be written as

$$\vartheta(X_t, S_t) = \frac{S_t^{1-\alpha}}{1-\alpha} \zeta(x_t),$$

where $\zeta(x_t) > 0$. Consider the optimization problem

$$\widehat{Q}(X_0, S_0; \tau_0) \equiv \max_{C, X_{0+}, S_{0+}, \phi_0} \frac{1}{1-\alpha} [h(\tau_0)]^\alpha C^{1-\alpha} + \beta^\tau E_0 \{\zeta(X_\tau, S_\tau)\}. \quad (101)$$

Using an identical argument to the one given above, shows that $\widehat{Q}(X_0, S_0; \tau_0)$ in equation (101) is concave if $\zeta(X_t, S_t)$ is concave. Moreover, \widehat{Q} is also homogenous of degree $1 - \alpha$ and hence can be rewritten as:

$$\widehat{Q}(X_0, S_0; \tau_0) = \frac{S_t^{1-\alpha}}{1-\alpha} q(x_t; \tau_0)$$

Next define

$$Q(X_0, S_0) = \max_{\tau_0} \widehat{Q}(X_0, S_0; \tau_0) \quad (102)$$

In order to show that the value function $V(X_0, S_0)$ is concave it suffices to show that Q is concave, by the argument in Stokey and Lucas (1989) (Corollary 1, page 52). Hence in the remainder of the proof, we focus on showing that Q is concave. Using the envelope theorem, one obtains $Q_X(X_0, S_0) = \widehat{Q}_X(X_0, S_0; \tau_0^*)$ and $Q_S(X_0, S_0) = \widehat{Q}_S(X_0, S_0; \tau_0^*)$ where τ_0^* is the value of τ_0 that maximizes (102).

Furthermore:

$$Q_X(X_0, S_0) = \widehat{Q}_X(X_0, S_0; \tau_0^*) = \frac{S_t^{1-\alpha}}{1-\alpha} q_X(x_t; \tau_0^*) = \frac{S_t^{-\alpha}}{1-\alpha} q_x(x_t; \tau_0^*) \quad (103)$$

$$Q_S(X_0, S_0) = \widehat{Q}_S(X_0, S_0; \tau_0^*) = S_t^{-\alpha} \left(q(x_t; \tau_0^*) - \frac{1}{1-\alpha} q_x(x_t; \tau_0^*) x \right). \quad (104)$$

The second-order derivatives are

$$Q_{XX} = \widehat{Q}_{XX} + \widehat{Q}_{X\tau_0} \frac{d\tau_0^*}{dX_t} \quad (105)$$

$$Q_{SS} = \widehat{Q}_{SS} + \widehat{Q}_{S\tau_0} \frac{d\tau_0^*}{dS_t} \quad (106)$$

$$Q_{SX} = \widehat{Q}_{SX} + \widehat{Q}_{S\tau_0} \frac{d\tau_0^*}{dX_t} = \widehat{Q}_{XS} + \widehat{Q}_{X\tau_0} \frac{d\tau_0^*}{dS_t} \quad (107)$$

Furthermore, the homogeneity of Q implies that τ_0^* is a function of x_t only. Hence there will exist some function $y(x_t)$ such that $\tau_0^* = y(x_t)$. This implies that

$$\frac{d\tau_0^*}{dX_t} = \frac{y'(x_t)}{S_t} \quad \text{and} \quad \frac{d\tau_0^*}{dS_t} = -\frac{y'(x_t)}{S_t} x_t.$$

Hence

$$\frac{\frac{d\tau_0^*}{dX_t}}{\frac{d\tau_0^*}{dS_t}} = -\frac{1}{x_t}. \quad (108)$$

Furthermore, since $\widehat{Q}_{SX} = \widehat{Q}_{XS}$ and $Q_{SX} = Q_{XS}$ we obtain

$$\widehat{Q}_{S\tau_0} \frac{d\tau_0^*}{dX_t} = \widehat{Q}_{X\tau_0} \frac{d\tau_0^*}{dS_t}. \quad (109)$$

Combining (108) and (109) gives

$$\frac{\frac{d\tau_0^*}{dX_t}}{\frac{d\tau_0^*}{dS_t}} = \frac{\widehat{Q}_{X\tau_0}}{\widehat{Q}_{S\tau_0}} = -\frac{1}{x_t} \quad (110)$$

Q_{XX}, Q_{SS}

To establish that Q is concave, it suffices to show that $Q_{XX}Q_{SS} - Q_{SX}^2 < 0$. Using (105)-(107) leads to

$$Q_{XX}Q_{SS} - Q_{SX}^2 = \left(\widehat{Q}_{XX} + \widehat{Q}_{X\tau_0} \frac{d\tau_0^*}{dX_t} \right) \left(\widehat{Q}_{SS} + \widehat{Q}_{S\tau_0} \frac{d\tau_0^*}{dS_t} \right) - \left(\widehat{Q}_{SX} + \widehat{Q}_{S\tau_0} \frac{d\tau_0^*}{dX_t} \right)^2 \quad (111)$$

The first term on the right hand side of (111) can be expressed as

$$\begin{aligned}
& \left(\widehat{Q}_{XX} + \widehat{Q}_{X\tau_0} \frac{d\tau_0^*}{dX_t} \right) \left(\widehat{Q}_{SS} + \widehat{Q}_{S\tau_0} \frac{d\tau_0^*}{dS_t} \right) = \\
& = \widehat{Q}_{XX} \widehat{Q}_{SS} + \widehat{Q}_{XX} \widehat{Q}_{S\tau_0} \frac{d\tau_0^*}{dS_t} + \widehat{Q}_{X\tau_0} \frac{d\tau_0^*}{dX_t} \widehat{Q}_{SS} + \left(\widehat{Q}_{X\tau_0} \frac{d\tau_0^*}{dX_t} \right) \left(\widehat{Q}_{S\tau_0} \frac{d\tau_0^*}{dS_t} \right) \quad (112) \\
& = \widehat{Q}_{XX} \widehat{Q}_{SS} + \frac{d\tau_0^*}{dS_t} \left[\widehat{Q}_{XX} \widehat{Q}_{S\tau_0} - \frac{1}{x_t} \widehat{Q}_{X\tau_0} \widehat{Q}_{SS} \right] + \frac{1}{x_t^2} \left(\widehat{Q}_{S\tau_0}^2 \right) \left(\frac{d\tau_0^*}{dS_t} \right)^2 \\
& = \widehat{Q}_{XX} \widehat{Q}_{SS} + \frac{d\tau_0^*}{dS_t} \widehat{Q}_{S\tau_0} \left[\widehat{Q}_{XX} + \frac{1}{x_t^2} \widehat{Q}_{SS} \right] + \frac{1}{x_t^2} \left(\widehat{Q}_{S\tau_0}^2 \right) \left(\frac{d\tau_0^*}{dS_t} \right)^2
\end{aligned}$$

where we have made repeated use of (110). The second term on the right hand side of (111) can be rewritten as

$$\begin{aligned}
\left(\widehat{Q}_{SX} + \widehat{Q}_{S\tau_0} \frac{d\tau_0^*}{dX_t} \right)^2 & = \widehat{Q}_{SX}^2 + 2\widehat{Q}_{SX} \widehat{Q}_{S\tau_0} \frac{d\tau_0^*}{dX_t} + \left(\widehat{Q}_{S\tau_0} \frac{d\tau_0^*}{dX_t} \right)^2 \quad (113) \\
& = \widehat{Q}_{SX}^2 - \frac{2}{x_t} \widehat{Q}_{SX} \widehat{Q}_{S\tau_0} \frac{d\tau_0^*}{dS_t} + \frac{1}{x_t^2} \left(\widehat{Q}_{S\tau_0}^2 \right) \left(\frac{d\tau_0^*}{dS_t} \right)^2.
\end{aligned}$$

where once again we have used (110) to arrive from the first line to the second.

Combining (??) and (113) gives:

$$Q_{XX}Q_{SS} - Q_{SX}^2 = \widehat{Q}_{XX}\widehat{Q}_{SS} - \widehat{Q}_{SX}^2 \quad (114)$$

$$+ \left(\widehat{Q}_{XX} + \frac{1}{x_t^2} \widehat{Q}_{SS} + \frac{2}{x_t} \widehat{Q}_{SX} \right) \widehat{Q}_{S\tau_0} \frac{d\tau_0^*}{dS} \quad (115)$$

The term $\widehat{Q}_{XX}\widehat{Q}_{SS} - \widehat{Q}_{SX}^2$ is non-positive by the concavity of \widehat{Q} . To determine the sign of the term $\widehat{Q}_{S\tau_0} \frac{d\tau_0^*}{dS}$, note that the first-order condition for τ_0^* is:

$$\widehat{Q}_{\tau_0}(S_0, X_0; \tau_0^*) = 0.$$

Differentiating this equation with respect to S_0 gives:

$$\widehat{Q}_{\tau_0 S}(S_0, X_0; \tau_0^*) + \widehat{Q}_{\tau_0 \tau_0}(S_0, X_0; \tau_0^*) \frac{d\tau_0^*}{dS} = 0$$

and therefore

$$\widehat{Q}_{S\tau_0} \frac{d\tau_0^*}{dS} = -\widehat{Q}_{\tau_0 \tau_0}(S_0, X_0; \tau_0^*) \left(\frac{d\tau_0^*}{dS} \right)^2 \geq 0,$$

where the inequality follows from $\widehat{Q}_{\tau_0 \tau_0}(S_0, X_0; \tau_0^*) \leq 0$.

Hence to establish that Q is concave it suffices to show that $\widehat{Q}_{XX} + \frac{1}{x_t^2} \widehat{Q}_{SS} + \frac{2}{x_t} \widehat{Q}_{SX} \leq$

0. Equations (104) and (103) imply after several simplifications

$$\begin{aligned}\widehat{Q}_{XS} &= -\alpha \frac{S_t^{-\alpha-1}}{1-\alpha} q_x(x_t; \tau_0^*) - \frac{S_t^{-\alpha-1}}{1-\alpha} x_t q_{xx}(x_t; \tau_0^*) = \frac{S_t^{-\alpha-1}}{1-\alpha} q_x(x_t; \tau_0^*) \left[-\alpha - \frac{x_t q_{xx}(x_t; \tau_0^*)}{q_x(x_t; \tau_0^*)} \right] \\ \widehat{Q}_{XX} &= \frac{S_t^{-\alpha-1}}{1-\alpha} q_{xx}(x_t; \tau_0^*) \\ \widehat{Q}_{SS} &= -\alpha S_t^{-\alpha-1} q(x_t; \tau_0^*) + 2\alpha \frac{S_t^{-\alpha-1}}{1-\alpha} x_t q_x(x_t; \tau_0^*) + \frac{S_t^{-\alpha-1}}{1-\alpha} x_t^2 q_{xx}(x_t; \tau_0^*)\end{aligned}$$

Combining the above three equations gives:

$$\begin{aligned}\widehat{Q}_{XX} + \frac{1}{x_t^2} \widehat{Q}_{SS} + \frac{2}{x_t} \widehat{Q}_{SX} &= \frac{S_t^{-\alpha-1}}{1-\alpha} q_{xx}(x_t; \tau_0^*) \\ &+ \frac{1}{x_t^2} \left(-\alpha S_t^{-\alpha-1} q(x_t; \tau_0^*) + 2\alpha \frac{S_t^{-\alpha-1}}{1-\alpha} q_x(x_t; \tau_0^*) x_t + \frac{S_t^{-\alpha-1}}{1-\alpha} q_{xx}(x_t; \tau_0^*) x_t^2 \right) \\ &+ \frac{2}{x_t} \left(-\alpha \frac{S_t^{-\alpha-1}}{1-\alpha} q_x(x_t; \tau_0^*) - \frac{S_t^{-\alpha-1}}{1-\alpha} q_{xx}(x_t; \tau_0^*) x_t \right) \\ &= -\alpha \frac{S_t^{-\alpha-1}}{x_t^2} q(x_t; \tau_0^*) < 0.\end{aligned}$$

This concludes the proof.

Proof of Lemma 7: Suppose that time t_j is an observation date and that (X_{t_j}, S_{t_j}) is in Region I. Therefore, the transactions balance at the next observation date is $X_{t_j+\tau} = 0$. Use the first-order conditions with respect to C and ΔS^{sell} in equations (74) and (75), respectively, to obtain $(1 - \psi_s) U'(C; \tau) = \beta^\tau E_{t_j} \{V_S(X_{t_j+\tau}, S_{t_j+\tau})\} (R^f)^\tau - \lambda^{\text{sell}}$. Rewrite this equation using (1) $C = X_{t_j^+} = \pi_1 S_{t_j^+}$ so equation (15) implies $U'(C; \tau) = [h(\tau)]^\alpha [\pi_1 S_{t_j^+}]^{-\alpha}$; (2) equation (18) which implies that $V_S(X_{t_j+\tau}, S_{t_j+\tau}) = V_S(0, S_{t_j+\tau}) = S_{t_j+\tau}^{-\alpha} v(0) = \left(R(t_j, t_j + \tau) S_{t_j^+}\right)^{-\alpha} v(0)$ so equation (68) implies $E_{t_j} \{V_S(X_{t_j+\tau}, S_{t_j+\tau})\} (R^f)^\tau = E_{t_j} \{V_S(X_{t_j+\tau}, S_{t_j+\tau}) R(t_j, t_j + \tau)\} = E_{t_j} \left\{ [R(t_j, t_j + \tau)]^{1-\alpha} S_{t_j^+}^{-\alpha} \right\} v(0)$; and (3) in Region I, $\Delta S^{\text{sell}} > 0$, so $\lambda^{\text{sell}} = 0$ to obtain $(1 - \psi_s) [h(\tau)]^\alpha [\pi_1(\tau)]^{-\alpha} = \beta^\tau E_{t_j} \left\{ [R(t_j, t_j + \tau)]^{1-\alpha} \right\} v(0)$. Use the definition of $J(\tau)$ in equation (80) and rearrange this equation to obtain $(1 - \psi_s) [h(\tau)]^\alpha [\pi_1(\tau)]^{-\alpha} = J(\tau) v(0)$, which implies $\pi_1(\tau) = \left(\frac{1-\psi_s}{v(0)}\right)^{\frac{1}{\alpha}} h(\tau) [J(\tau)]^{-\frac{1}{\alpha}}$. q.e.d.

Proof of Lemma 8: Suppose that time t_j is an observation date, and at time t_j the transactions account has a zero balance, so that $x_{t_j} = 0$. Then equation (46) implies that $S_{t_j^+} = \frac{1-\psi_s}{1-\psi_s+\pi_1(\tau)} S_{t_j}$ and equation (44) implies that $X_{t_j^+} = \pi_1(\tau) S_{t_j^+}$. Lemma 4 implies that for $x_{t_j} = 0$, $C = X_{t_j^+} = \pi_1(\tau) S_{t_j^+}$. The next observation date is $t_j + \tau$, and on this observation date $X_{t_j+\tau} = 0$ and $S_{t_j+\tau} = R(0, \tau) S_{t_j^+}$. Therefore, the value function in equation (26) can be written

as $\widehat{V}(0, S_{t_j}; \tau) = \frac{1}{1-\alpha} [h(\tau)]^\alpha \left(\pi_1 \frac{1-\psi_s}{1-\psi_s+\pi_1(\tau)} S_{t_j} \right)^{1-\alpha} + \beta^\tau E_{t_j} \left\{ V \left(0, R(t_j, t_j + \tau) \frac{1-\psi_s}{1-\psi_s+\pi_1(\tau)} S_{t_j} \right) \right\}$. Use equation (28) for $\widehat{V}(X_{t_j}, S_{t_j}; \tau)$ and equation (18) for $V(X_{t_j}, S_{t_j})$ to rewrite the conditional value function as $\widehat{v}(0; \tau) = \left(\frac{1-\psi_s}{1-\psi_s+\pi_1(\tau)} \right)^{1-\alpha} \left[[h(\tau)]^\alpha [\pi_1(\tau)]^{1-\alpha} + \beta^\tau E_{t_j} \left\{ [R(t_j, t_j + \tau)]^{1-\alpha} \right\} v(0) \right]$. Use the definition of $J(\tau)$ in equation (80) and use Lemma 7 to rewrite the conditional value function as $\widehat{v}(0; \tau) = \left(\frac{1-\psi_s}{1-\psi_s+\pi_1(\tau)} \right)^{1-\alpha} \left[\frac{1}{1-\psi_s} \pi_1(\tau) v(0) J(\tau) + J(\tau) v(0) \right] = \left(\frac{1-\psi_s}{1-\psi_s+\pi_1(\tau)} \right)^{-\alpha} J(\tau) v(0) = \left(1 + \frac{\pi_1(\tau)}{1-\psi_s} \right)^\alpha J(\tau) v(0)$. q.e.d.

Proof of Proposition 6: To find the optimal value of τ , first differentiate the conditional value function $\widehat{v}(0; \tau)$ in Lemma 8 with respect to τ and set the derivative equal to zero to obtain $\alpha \frac{\pi_1(\tau)}{1-\psi_s} \frac{\pi_1'(\tau)}{\pi_1(\tau)} = - \left(1 + \frac{\pi_1(\tau)}{1-\psi_s} \right) \frac{J'(\tau)}{J(\tau)}$. Differentiate the expression for $\pi_1(\tau)$ in Lemma 7 with respect to τ to obtain $\frac{\pi_1'(\tau)}{\pi_1(\tau)} = \frac{h'(\tau)}{h(\tau)} - \frac{1}{\alpha} \frac{J'(\tau)}{J(\tau)}$ and substitute this expression for $\frac{\pi_1'(\tau)}{\pi_1(\tau)}$ into the preceding expression to obtain $\frac{\pi_1(\tau)}{1-\psi_s} \frac{h'(\tau)}{h(\tau)} = -\frac{1}{\alpha} \frac{J'(\tau)}{J(\tau)}$. Evaluate this equation at $\tau = \tau^*$ and use Proposition 5 to obtain $\frac{\pi_1(\tau)}{1-\psi_s} \frac{h'(\tau)}{h(\tau)} = -\frac{1}{\alpha} \frac{1}{[J(\tau^*)]^{-\frac{1}{\alpha}-1}} \frac{J'(\tau)}{J(\tau)}$.

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