Small and large price changes and the propagation of monetary shocks *

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Abstract

We document the presence of both small and large price changes in individual price records from the CPI in France and the US. After correcting for measurement error and cross-section heterogeneity we find that the size distribution of price changes has a positive excess kurtosis, with a shape that lies between a Normal and a Laplace distribution. We propose a model, featuring random menu-costs and multi product firms, that is capable to reproduce the observed empirical patterns. We characterize analytically the response of the aggregate economy to a monetary shock. Different propagation mechanism, spanning the models of Taylor (1980), Calvo (1983) and Golosov and Lucas (2007), are nested under different combinations of four fundamental parameters. We discuss the identification of these parameters using data on the size-distribution of price changes and the actual cost of price adjustments borne by firms. The output effect is proportional to the ratio of kurtosis to the frequency of price changes.

JEL Classification Numbers: E3, E5

Key Words: price setting, micro evidence, size-distribution of price changes, kurtosis of price changes, menu-cost, Calvo pricing rule, output response to monetary shocks

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1 Introduction

This paper uses new micro evidence, and a new menu-cost model, to study the propagation of monetary shocks in an economy with sticky prices. By combining the assumptions of multiproduct firms and random menu costs the model is able to account for the coexistence of small and large price changes we observe in the micro data. Different set-ups with price rigidity, spanning the models of Taylor (1980), Calvo (1983), Reis (2006), Golosov and Lucas (2007), some versions of the "CalvoPlus" model by Nakamura and Steinsson (2010), as well as the multi-product models of Midrigan (2011), and Alvarez and Lippi (2013b), are nested by our model. This unified framework allows us to unveil which assumptions are required to obtain each of them as an optimal mechanism. Further, it allows us to compare the monetary transmission mechanism in the different models analytically. We argue that the question is relevant: the total cumulative output effect of a monetary shock in a Calvo setup is about 6 times larger than in the Golosov-Lucas model. Our model suggests that simple summary statistics, measurable from the distribution of price changes and from the cost of price adjustments, are key to select among these models.

Our contribution delivers new results on the empirics and the theory of sticky prices, and may be summarized as follows. The empirical contribution documents the presence of small and large price changes, i.e. a very peaked distribution of price changes, using a dataset of price records underlying the French CPI. This finding persists even at a very disaggregate level of product-outlet-type, ruling out an explanation based on pure cross-section heterogeneity, and it is similar to what Klenow and Kryvtsov (2008) detect for the US. We also acknowledge that the CPI data may contain measurement error that tends to distort the measure of peakedness of the distribution of price changes, an issue emphasized by Eichenbaum et al. (2013). We estimate that –after taking into consideration measurement error and cross-sectional heterogeneity– the shape of the size-distribution of price changes is in between a Normal and a Laplace distribution, with a kurtosis that is about 4 for the US and about 5 for France.

We develop an analytical model that matches these patterns qualitatively, featuring both the small and large price changes which lead to excess kurtosis. The model uses the multi-product setup developed in Alvarez and Lippi (2013b), where the fixed menu cost applies to a bundle of n goods. This assumption generates the small price changes. We extend that setup by introducing random menu costs, a feature that allows the model to replicate the positive excess kurtosis of price changes. In particular, we assume that with an exogenous rate λ a firm receives an opportunity to adjust its price at no cost, as in a Calvo setup. The model has four fundamental parameters: the size of the fixed cost relative to curvature of the profit function ψ/B, the volatility of idiosyncratic cost shocks σ^2, the number of products n and the arrival rate of free adjustments λ. These parameters completely determine the steady state statistics, such as the shape of the distribution of price changes (and hence its kurtosis), its standard deviation $Std(\Delta p_i)$, and the frequency of price changes, N_a. They are also key to determine the response to a monetary shock.

The model yields several new theoretical results. First we characterize how the inaction set behaves as function of parameters. For a small menu cost the model behaves as in Barro (1972); Dixit (1991); Golosov and Lucas (2007): the size of the inaction set displays the usual high sensitivity (i.e. a "quartic root") with respect to the cost and the volatility of the shocks σ^2 (the option value effect). Interestingly, the decision rule is unaffected by the presence of the free adjustments as long as their arrival rate λ is small. The decision rule changes substantially for large menu costs ψ/B, an assumption that is useful to generate behavior that approaches that of a Calvo model. In this case the size of the inaction set changes with the square root of the menu cost and the arrival rate, and somewhat surprisingly it becomes unresponsive to the volatility of idiosyncratic shock σ^2, so that changes in the uncertainty faced by firms induce no change in behavior (i.e. there is no option value). Another novel result is a mapping between the costs of price adjustment and the parameters of the model: we give a complete analytical characterization of the menu cost implied by observable statistics such as the frequency and the variance of price changes, as well as others. This mapping can be used to quantify a value of ψ consistent with the evidence on the costs of price adjustment, as measured by e.g. Levy et al. (1997), or it can be used to assess the plausibility of benchmark models such as Calvo pricing.

Second, by aggregating the optimal decision rules across firms we characterize the frequency N_a, size $Std(\Delta p_i)$, and shape of the distribution of the price changes. Some combinations of the fundamental parameters affect N_a and $Std(\Delta p_i)$ without changing the shape of the distribution of price changes. Alternatively, some (combinations) of the fundamental parameters affect the shape of the distribution of price changes, while keeping N_a and $Std(\Delta p_i)$ fixed. We show that the shape of the distribution of price changes can be written exclusively in terms of n and the fraction of free-adjustments $\ell \equiv \lambda/N_a$, a re-parameterization we find quite intuitive. The shape ranges from bimodal (for the model where $\ell = 0$ and $n = 1$ as in the Golosov-Lucas model) to Normal (for $n = \infty$ and $\ell = 0$, our version of Taylor or Reis (2006)'s model), and up to Laplace (in the case $\ell = 1$ for any n, our version of the Calvo model). In those three models, the kurtosis of price changes is, respectively, 1, 3 and 6. In general for any given ℓ the level of kurtosis is increasing in n. Likewise, for a given n the level of kurtosis is increasing in ℓ.[1] A large kurtosis of a symmetric random variable is due to

[1]In our set-up a given kurtosis may be obtained by different combinations of n and ℓ, yet we argue that

the simultaneous presence of small and large realizations, relative to intermediate values. In our model a large kurtosis of price changes is produced by the following mechanism: small price changes occur due to the multi product nature of adjustment and/or due to the arrival of a free adjustment opportunity. For large price change to occur it is necessary to have a large adjustment cost ψ/B, otherwise a large deviation from the ideal price would not be tolerated.

Third, we use the model to solve analytically for the impulse responses of the aggregate economy to a once-and-for-all unexpected permanent increase of the money supply. The aggregate effect of a monetary shock depends on the shock size, the frequency N_a, the size (e.g. $Std(\Delta p_i)$) and the shape (e.g. kurtosis) of price adjustments. The dependence on the size of the shock is a hallmark of menu cost models: monetary shocks that are large (relative to the size of price adjustments) lead to almost all firms adjusting prices and hence imply neutrality. The dependence on the frequency N_a is intuitive: model economies with the same distribution of price changes, but more price changes per year, return to the flexible price with a velocity that is proportional to N_a. Lastly, and new in the literature, we isolate the dependence between the features that determine the *shape* of the distribution of price changes and the effect of a (small) monetary shock. Surprisingly, fixing the frequency N_a and scale $Std(\Delta p_i)$ of price adjustments, the real effects of monetary policy are an increasing function of *only one parameter*: the kurtosis of the steady-state distribution of price changes. We show that the cumulated response of real output following a monetary shock is proportional to kurtosis, whose value ranges between 1 to 6 as combinations of n and ℓ span different models. Such a result has potential empirical implications since, after taking into account measurement error, the kurtosis of the price changes can be measured from available data. Furthermore, our theoretical decomposition of the determinants of the area under the impulse response shows the way to measure and aggregate across heterogenous sectors, simply by measuring the kurtosis of standardized price changes, and aggregating the sectorial effects with weights that are inversely proportional to the frequency of price changes.

Related literature

Our paper relates to a large literature on the propagation of monetary shocks in sticky price models, unifying earlier results that compare the propagation in the Calvo model with the propagation in either the Taylor or the menu cost model of sticky prices.[2] Interestingly, we

models with high n yield a better representation of the cross-sectional data.

[2]A selected list is Kiley (2002), Caballero and Engel (2007), Golosov and Lucas (2007). Kiley (2002) obtains that, controlling for the frequency of price changes, the response of output is stronger and more persistent under Calvo than under Taylor contracts. Golosov and Lucas (2007) compare a monetary shock in a menu cost and a Calvo model, with similar frequencies of price change, and find that the half-life of the

show that introducing the random adjustment costs serves a similar role as that of fat-tailed shocks in Midrigan (2011), increasing the real effect of monetary shocks and bringing the model behavior closer to a Calvo model (yet there are some difference, see Appendix F for a formal discussion).

The special case of one product ($n = 1$) of our model is related to the seminal work by Dotsey, King, and Wolman (1999) on the propagation of shocks when firms face a random menu cost. For our purposes it is important to simultaneously consider idiosyncratic cost shocks, as done in Dotsey, King, and Wolman (2009) and Nakamura and Steinsson (2010). Relative to Nakamura and Steinsson's (2010), whose CalvoPlus model is similar to ours, we abstract from more general monetary policies and from the linkages between sectors. Also relative to Dotsey, King, and Wolman's (2009) we abstract from capital accumulation and aggregate productivity shocks.[3] We view our work as complementary to theirs. Our model allows for an analytical characterization of the firm's decision rule, the economy's steady state statistics, the identification of the key model parameters, as well as a characterization of the relationship between these parameters/statistics and the size of the output effect of monetary shocks. We think that our approach not only clarifies the different forces at play, but allows us to analyze substantive issues. Among those issues are (i) the plausibility of the menu cost required to have (almost) Calvo-pricing, (ii) what features of the economy determine the output effect of monetary shocks, (iii) defining the proper statistics (such as kurtosis and the frequency of price changes) that allow us to quantify (ii).

The paper is organized as follows: the next section presents the cross section evidence on price setting behavior using data for France and the USA taken from various sources. Section 3 presents the theoretical model and its cross section predictions: it is shown that the model has fundamentally four parameters and we discuss the mapping between those and observable measures of price setting behavior. Section 4 derives the model predictions on the effect of an unexpected monetary shock. Section 5 summarizes the contribution of the paper and discusses some implications for quantifying the real effects of monetary shocks.

2 The distribution of price changes: micro-evidence

A vast amount of research has investigated the patterns of price changes at the microeconomic level in the past decade. A recurring fact that emerges from those studies is that the size distribution of price changes exhibits a large amount of small price changes, as noted by Klenow and Malin (2010); Cavallo (2010); Klenow and Kryvtsov (2008); Chen et al. (2008)

response to a monetary policy shock is about five times larger in a Calvo set-up.

[3]See Section 3 and Appendix B for a comparison with previous models on stochastic menu costs.

and Midrigan (2009, 2011) using selected samples of micro data from the US as well as many other industrial countries. This section revisits this evidence using a detailed dataset of price quotes underlying the French Consumer Price Index (about 65% of the CPI weights from 2003 to 2011). We discuss measurement error by comparing the CPI data with other sources, presumably less affected by measurement error. Finally, we compare our evidence with comparable evidence for the US.

Two issues that are investigated in detail concern heterogeneity and measurement error. Heterogeneity across type of goods and of outlets is pervasive in price data. A well known result is that a mixture of distributions with different variance and the same kurtosis will have a larger kurtosis. For this reason we standardize the data at levels at which we suspect there is heterogeneity in the variances and focus on the kurtosis of the pooled data. We define the standardized price changes, z, by demeaning and dividing by the standard deviation of price changes at fine cell levels. A cell is a category of good and of outlet type. We then compute the statistics for the pooled standardized data.[4] The nature of the correction for measurement error is to compare the CPI statistics with data for similar goods and outlet types that are less affected by measurement error, as in the internet store scraped data from Cavallo (2010), and the scanner data sets used by Midrigan (2011), Eichenbaum et al. (2013) among others. We also analyze the effect of outliers by looking at the differential effect trimming across datasets. The practice of normalizing the data as well as removing outliers has been used before, as in e.g. Klenow and Kryvtsov (2008); Midrigan (2009).

We find it useful to compare the empirical distribution of price changes to three parametric distributions ordered in terms of increasing frequency of "extreme" price changes: the binomial, the Normal, and the Laplace distribution. Our analysis shows that, after removing the (time invariant) cross section heterogeneity and correcting for measurement error, the size distribution of price changes still features a considerable mass of large as well as small price changes, relative to the binomial distribution implied by the standard menu cost model. Overall we conclude that, after taking into account heterogeneity and measurement error, the shape of the empirical distribution of price changes lays in between a Normal and a Laplace distribution. To quantify the presence of extreme price changes we focus on statistics that are informative about the shape of the size distribution, that are appropriate for symmetric, zero-mean, distributions and that are scale-free. These statistics measure the frequency of extreme (i.e. large and small) observations relative to the standard deviation of the distribution. Because of its prominent role in the theoretical analysis we will focus especially on kurtosis whose value, for the benchmark Binomial, Normal and Laplace distribution, is 1, 3 and 6 respectively.

[4] The model of Section 3 allows for sector (and/or good or outlet) heterogeneity and discusses aggregation.

2.1 The French microeconomic data underlying the CPI

The data are a longitudinal dataset of monthly price quotes collected by the INSEE in order to compute the French CPI, over the period 2003:4 to 2011:4.[5] Each record relates to a precisely defined product sold in a particular outlet in a given year and month. It contains the price level of the product, as well as limited additional information such as an outlet identifier, an index (when relevant) for package size (say 1 liter) and flags indicating the presence of sales. The raw dataset contains around 11 million price quotes and covers about 65% of the CPI weights.[6] The dataset also includes CPI weights, which we use to compute aggregate statistics. Price changes are computed as 100 times the log-difference in prices per unit. To minimize the presence of measurement errors we discarded observations with item substitutions (which might give rise to spurious price changes) and removed "outliers" which, in our baseline analysis, we defined as price changes smaller than 0.1 percent, or larger than $100 \log(10/3)$ (both in absolute value).[7]

An important issue with the data on price changes is the treatment of sales. The relevance of dealing with sales in analyzing price stickiness was emphasized by Nakamura and Steinsson (2008); Kehoe and Midrigan (2007) and Midrigan (2011) inter alia. The INSEE dataset contains an indicator variable that identifies whether a given observed price corresponds to a sales promotion discount (either seasonal sale or temporary discounts).[8] Price changes that result from sales (including price changes from a sales price to a regular one) account for approximately 17% of all the price changes. Overall, the incidence of sales on the frequency of price change is less important than in the US where according to Nakamura and Steinsson (2008) the share of price change due to sales is 21.5%. In the following, as a robustness check, we report results both with and without sales observations.

We now document the patterns on the peakedness and thick tails of the distribution of price changes. As those patterns vary considerably across sectors and outlet type, a concern already mentioned is that a large variance and kurtosis of price changes may essentially reflect that observations of price changes are drawn from a mixture of distributions, and thus be artefacts.[9] In what follows we address this concern by considering the distribution

[5]The dataset is documented in details in Berardi, Gautier, and Le Bihan (2013).

[6]Some categories of goods and services are not available in our sample: fresh foods, rents, and prices centrally collected by the statistical institute - among which car prices and administered and public utility prices (e.g. electricity). Note that, while rents are out of our dataset, cost of owner-occupied housing is not incorporated in the French CPI, so the share of housing is the CPI is lower than in some other countries.

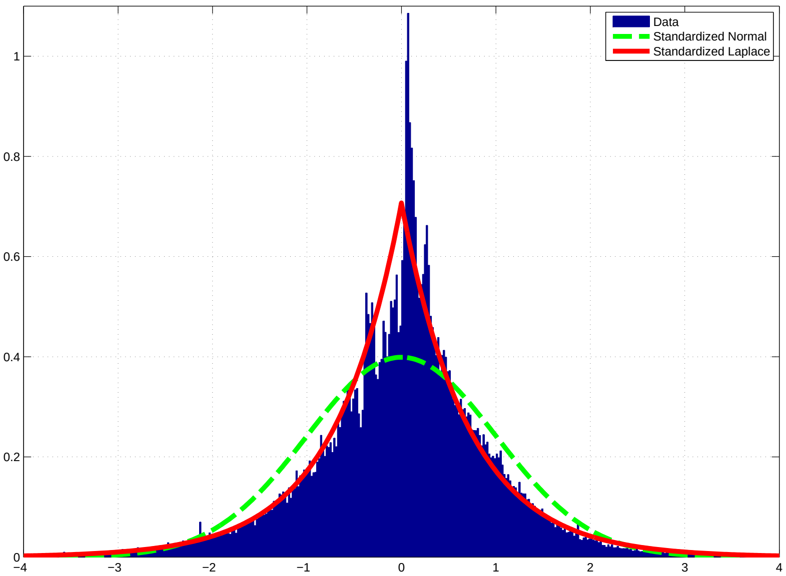
[7]Some sales involve large discounts, up to 70%, e.g. in clothing. The upper threshold $100 \log(10/3)$ allows to accommodate the after-sale price to return to the original level. See Appendix C for more information and several robustness checks.

[8]The flag is documented by the field agent rather than constructed using a statistical filter. Baudry et al. (2007) investigate the extent of "undetected" sales and conclude this is a limited concern.

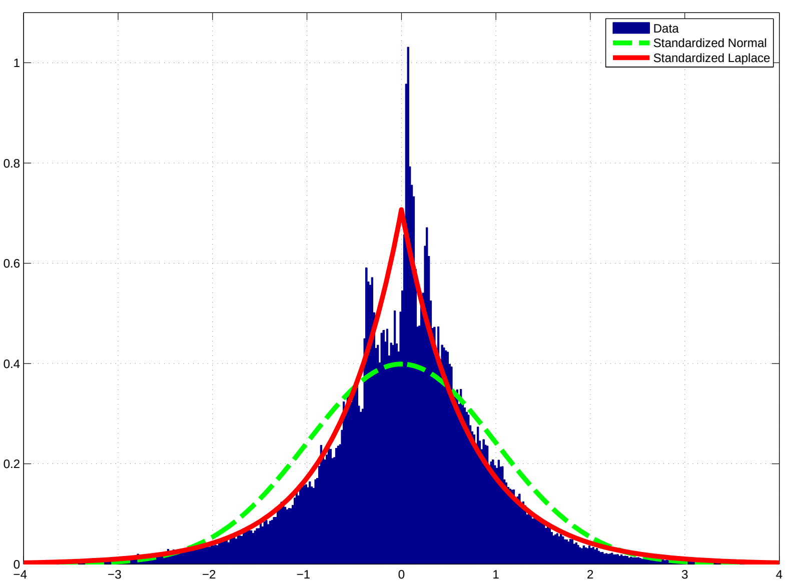
[9] Formally, let Δp be a mixture of the $i = 1, ..., M$ distributions Δp_i with strictly positive weights, where

Figure 1: Histogram of Standardized Price Adjustments: French CPI 2003-2011

All data



Excluding sales



The figures use the elementary CPI data from France (2003-2011). Price changes are the log difference in price per unit, standardized by good category (272) and outlet type (11) and pooled. Price changes equal to zero are discarded. The upper panel uses about 1.5 million data points, the lower panel about 1.1 million.

of standardized price changes.[10] We consider a breakdown of the data into J "cells" (for instance, one cell will be bread in supermarkets). In each cell j the standardized price change for an item i at date t is defined as $z_{ijt} = (\Delta p_{ijt} - m_j)/\sigma_j$ where m_j and σ_j are the mean and standard deviation of price changes in cell j, and price changes equal to zero are disregarded. We will here use the finest partition possible in our data (each cell is a COICOP category at the 6-digit level in an outlet type) and have around 1,500 cells.[11] Figure 1 is a weighted histogram of the standardized price changes. On the same graph we superimpose the density of the standard Normal distribution as well as the standardized Laplace distribution (both have unit variance). The Laplace distribution has a kurtosis of 6 and is thus more peaked than the Normal. It is apparent that the empirical distribution of standardized price changes is closer to the Laplace distribution than to the Normal.[12] We also consider the statistic $\mathbb{E}[|\Delta p_i|]/Std(\Delta p_i)$ as a (reverse) measure of the frequency of extreme price changes. The main difference of this statistic with respect to kurtosis is that it is less sensitive to extreme outliers. For the Binomial, Normal and Laplace distributions the reference values for this statistic are: 1, 0.80 and 0.70.

Table 1 reports the frequency of price changes as well as selected moments of the distribution of price changes. The frequency of price change is around 17% per month. The fraction of price decreases among price changes is around 40%. The average *absolute* price change is sizeable (9.19%), as is the standard deviation of price change (16.6%). These patterns match those documented by Alvarez et al. (2006) for the Euro area. With the qualification that frequency of price change is typically found to be smaller in the Euro area than in the US, they also broadly match US evidence provided by e.g. Nakamura and Steinsson (2008). The kurtosis and peakedness of the distribution of price changes have not been quantitatively documented so far on European data. The kurtosis of non-standardized price changes is very high: 12.8. This level of kurtosis is of same order of magnitude as that documented by Klenow and Malin (2010) for the US. As argued above, a high kurtosis is likely the consequence of a mixture of observations taken from distributions with different variances. Considering the standardized price changes, i.e. correcting for cross-section heterogeneity in the variances,

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we assume that $E[\Delta p_1] = \cdots = E[\Delta p_N] = 0$. Denote by k_i the kurtosis $k_i = E[\Delta p_i^4]/(Std(\Delta p_i)^4)$ and by r_i the ratio $r_i = E[|\Delta p_i|]/Std(\Delta p_i)$ for each distribution i. Assume that $k_1 = k_2 = \cdots = k_M \equiv k$ and that $r_1 = r_2 = \cdots = r_M \equiv r$. Then the statistics for the pooled data satisfy: $E[\Delta p_i^4]/(Std(\Delta p_i)^4) \geq k$ and $E[|\Delta p_i|]/Std(\Delta p_i) \geq r$, with equality iff $Std(\Delta p_1) = Std(\Delta p_2) \cdots = Std(\Delta p_M)$. Thus, standardizing the M distributions will preserve the values of k and r.

[10] This follows Klenow and Kryvtsov (2008), as well as Midrigan (2011).

[11] There are 11 outlet types and 272 CPI categories. Not every category of good is sold in every outlet type so there are about half of the potential 2,992 cells.

[12] In an online appendix, we provide similar histograms by groups of good at a disaggregated level. Most of them have the same pattern as Figure 1, that is a distribution that is more peaked than the gaussian, and often more peaked than the Laplace.

Table 1: Selected moments from the distribution of price changes

	Data		Benchmarks	
	all records	exc.sales	Normal	Laplace
Frequency of price changes	17.09	14.70		
Fraction of price changes that are decreases	39.23	35.73		
Moments for the size of price changes: Δp				
Average	0.33	1.06		
Standard deviation	16.60	8.01		
Kurtosis	12.81	20.86		
Moments of standardized price changes: z				
Kurtosis	8.89	10.40	3	6
Moments for the absolute value of standardized price changes: $\|z\|$				
Average: $\mathbb{E}(\|z\|)$	0.70	0.69	0.80	0.70
Fraction of observations $< 0.25 \cdot \mathbb{E}(\|z\|)$	22.2	20.7	15.8	22.1
Fraction of observations $< 0.5 \cdot \mathbb{E}(\|z\|)$	39.3	38.6	31.0	39.4
Fraction of observations $> 2 \cdot \mathbb{E}(\|z\|)$	12.9	12.5	11.1	13.5
Fraction of observations $> 4 \cdot \mathbb{E}(\|z\|)$	1.8	2.0	0.0	1.8
Number of obs. with $\Delta p \neq 0$	1,544,829	1,080,183		

Source is INSEE, monthly price records from French CPI, data from 2003:4 to 2011:4. Coverage is around 65% of CPI weight since rents, and prices of fresh food and centrally collected items (e.g. electricity, train and airplane tickets) are not included in the dataset. Frequency of price change is the average fraction of price changes per month, in percent. Size of price change is the first-difference in the logarithm of price per unit, expressed in percent. Observations with imputed prices or quality change are discarded. Observations outside the interval $0.1 \leq |\Delta p| \leq 100 \cdot \log(10/3)$ are removed as outliers. "Exc. sales" exclude observations flagged as sales by the INSEE data collectors. Moments are computed aggregating all prices changes using CPI weights at the product level. The third and fourth panels report moments for the standardized price change $z_{ijt} = \frac{\Delta p_{ijt} - m_j}{\sigma_j}$ where m_j and σ_j are the mean and standard deviation of price changes in category j (see the text). The Normal and Laplace distributions used in the last two columns have a zero mean and standard deviation equal to one.

reduces kurtosis to 8.9 (also, kurtosis is halved on data excluding sales). Unfortunately the information available in the CPI data prevents us from correcting heterogeneity at a finer level (e.g., we do not know the UPC of the product or the store where it is sold). In other databases where more information is available the reduction in the measured kurtosis is even more prominent: Tables 8 and 9 in Labbé (2013), based on Nielsen scanner data for Chile, show that kurtosis falls by another 30 to 50% when moving to the product-store level.

The fraction of extreme (small or large) price changes is noticeable. The fraction of absolute standardized price changes lower than one fourth of the mean is 22.2 percent. Also 12.9 percent of absolute normalized price changes are larger than 2 times the mean of the absolute standardized price change. Overall, it appears that these figures are very close to the ones that would be produced by a (standardized) Laplace distribution. Consistently, the size of the average absolute standardized price change in the data is equal to 0.70, the same value that obtains for the statistic $\mathbb{E}[|\Delta p|]/Std(\Delta p)$ if Δp follows a Laplace distribution.

Removing sales has a large effect on the variance of absolute price change, as indicated by the results reported in the second column of Table 1.[13] However, removing sales does not affect our findings on the peakedness of the distribution. Kurtosis actually increases when sales observations are removed both in the raw data as well as in the standardized data. This is also visible in the bottom panel of Figure 1 which plots the distribution of standardized non sales-related price changes.

2.2 The magnitude of measurement error on estimates of kurtosis

In this section we discuss evidence on the magnitude of both very large and very small price changes due to measurement error, and discuss its effect on measures of kurtosis. Eichenbaum et al. (2013) warned that the small price changes recorded in the data may reflect measurement error. Appendix C.2 explores the concerns raised by Eichenbaum et al. (2013) and concludes that they they also apply to the French data, albeit to a lesser extent. We analyze here the consequences of one particular type of measurement error and derive a simple correction for kurtosis. We show that a small amount of this measurement error, inconsequential for measuring the aggregate the cost of living, may have sizeable consequences for the measurement of the descriptive statistics displayed in Table 1, such as kurtosis, and suggest a procedure to correct for it.

Let the observed price changes Δp_m be given by a mixture of two distributions:

$$\Delta p_m = \begin{cases} \Delta p_u & \text{with prob. } \zeta \\ \epsilon & \text{with prob. } 1 - \zeta \end{cases}$$

where we interpret ϵ as measurement error and Δp_u as a "true" price change. This assumption aims to capture that, even at the finest level of disaggregation, some price changes in the CPI data are the consequence of small product substitutions (e.g. different brands for a given good being recorded) which do not reflect an actual change in the good's price. Likewise, in scanner dataset, spuriously small price changes may originate from the weekly nature of the prices being recorded, which e.g. averages customers with and without discount coupons. Assume the distribution of Δp_u has standard deviation σ_u and kurtosis k_u. Likewise the distribution of ϵ has kurtosis k_e and standard deviation σ_e. Both distributions are assumed

[13]We remove the observations flagged as sales as well as the subsequent increase back to the "regular" price. To compute the standardized non-sales price changes, we first discard sales-related observations, then standardize the data.

to have zero expected value. One interpretation is that quality changes (not recorded by the statistical office) generate "artificial" price changes. We assume that these price changes are small, i.e. that σ_e is small, and that the process for the unreported changed in quality is independent of the "true" changes in prices. The kurtosis of the observed price changes is then equal to:

$$kurt[\Delta p_m] = k_u \, \frac{\zeta\sigma_u^4 + (k_\epsilon/k_u)\sigma_e^4}{\zeta^2\sigma_u^4 + (1-\zeta)^2\sigma_e^4 + 2\zeta(1-\zeta)\sigma_e^2\sigma_u^2}$$

Letting σ_e go to zero we obtain that kurtosis measured over the (observed) price changes is:

$$\lim_{\sigma_e \downarrow 0} kurt[\Delta p_m] = \frac{k_u}{\zeta} \tag{1}$$

Thus, if the sample includes a fraction ζ of true price changes and the rest are spuriously imputed small price changes the kurtosis will increase by a factor $1/\zeta$, relative to the kurtosis of the true distribution.[14] Equation (1) suggests to quantify ζ by comparing the observed kurtosis across a sample with measurement error and one without. We now turn to addressing this issue empirically.

Table 2: Comparison of the CPI vs. the BPP data in France

Statistic	BPP retailer 1	BPP retailer 5	CPI Hypermarkets	BPP retailer 4	CPI Large ret. electr.
duration	8.58	8.06	4.82	6.44	7.24
			Statistics for standardized price changes: z		
mean $\lvert z\rvert$	0.71	0.70	0.65	0.78	0.70
% below 0.5 mean $\lvert z\rvert$	37.85	40.93	45.48	29.17	41.69
% below 0.25 mean $\lvert z\rvert$	17.46	25.26	26.19	15.33	23.10
kurtosis of z	5.50	4.30	10.15	2.82	6.33

Note: The BBP data are documented in Cavallo (2010). Results were communicated by the author. For CPI data source is INSEE, monthly price records from French CPI, data from 2003:4 to 2011:4. Sub-sample in column (3) is price records in outlet type "hypermarkets". Sub-sample in column (5) is goods in the category of appliances and electronic , as identified using the Coicop nomenclature, collected in the following outlets type: "hypermarkets","supermarkets", and "large area specialists". Data are standardized within each subsample using Coicop categories.

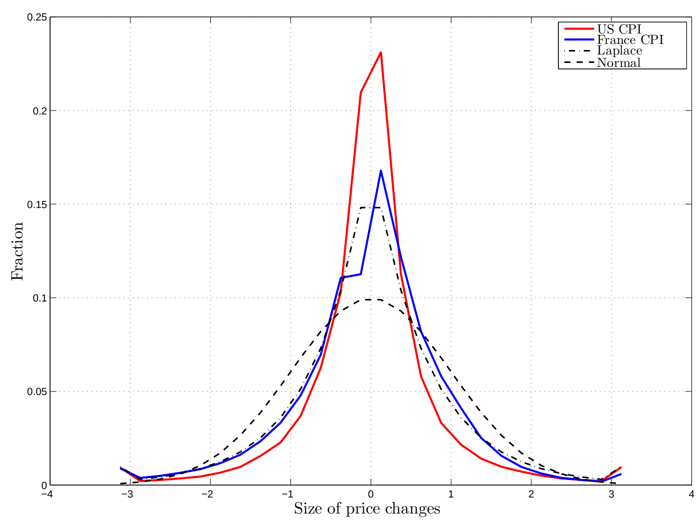
We match a subset of our French CPI data with the prices for several French retailers taken from the Billion Price Project (BPP) dataset (see Cavallo (2010)). The BPP data are

[14] Under this interpretation the number of measured price changes, denoted by N_{am} will be higher than the number of true price changes per unit of time, say N_{au}. Let's denote $N_{a\epsilon}$ the expected number of incorrectly imputed price changes. We have: $N_{am} = N_{au} + N_{a\epsilon} = \zeta N_{am} + (1-\zeta)N_{am}$. Thus if we have two estimates of $kurt[\Delta p_i]$ and of N_a and we assume that one has no measurement error and the other has a fraction ζ of small imputed price changes as described above, can estimate ζ using either the ratio of the two estimates of kurtosis or the ratio of the two estimates of the number of price changes per unit of time.

"scraped" on-line, thus they are arguably less contaminated by measurement errors.[15] We compare the results obtained using the scraped BPP data from two large retailers with our results based on the CPI data for a similar type of outlet: to this end we restrict our dataset to CPI price records in "hypermarkets", excluding gasoline. We also compare with the BBP data from a large French retailer specialized in electronic and appliances. In that case we restrict the CPI dataset to goods in the category of appliances and electronic using the Coicop nomenclature, collected in outlets type "hypermarkets","supermarkets", and "large area specialists".

Comparing the values of kurtosis from both data sets suggests that $\zeta \cong 0.5$. We can apply this magnitude to the full sample of CPI data of Table 1, for which no "measurement error-free" counterpart like the BPP exists, to obtain a corrected kurtosis. The number thus obtained for the kurtosis ranges between 4 and 5 (using the kurtosis of 8.89 of standardized price changes), so it lays in between the kurtosis of the Normal and the Laplace distribution.

Figure 2: Histogram of Standardized Price Adjustments: US and French CPI



Sales data are excluded. Data for France are from the CPI as in Figure 1. The CPI data for the US are taken from Figure 3 in Klenow and Kryvtsov (2008). Price changes equal to zero are discarded.

Sensitivity to trimming. To assess the hypothesis that large price changes are due to measurement error we compare the differential effect on kurtosis of trimming large (absolute value) price changes in CPI vs trimming them in scanner data. For the French CPI we find

[15] We are extremely grateful to Alberto Cavallo for sharing part of his data with us.

that excluding (log percent) price changes for which $|\Delta p| \geq 100\log(10/3)$, instead of just excluding those that are $|\Delta p| \geq 100\log(2)$, decreases kurtosis from 8.89 to 7.21 (see lines 4 and 8 of Table 7). We interpret this as evidence of measurement error under the hypothesis that large price changes in the CPI are due to transcription data errors, which are virtually absent in scanner data.[16]

For completeness we note that several (but not all) scanner price data are likely to be affected by measurement error since they report average weekly prices for an item and/or they report the mixed of prices paid by customers that may obtain discounts. We hypothesize, as Midrigan (2011) and Eichenbaum et al. (2013) among others in the literature, that this introduces spurious small price changes. Consistent with this view, we find that for the scanner data described in Table 4 excluding (log percent) price changes for which $|\Delta p| \leq 1/10$ reduces kurtosis from 10 to about 3. An effect on the same direction although of smaller magnitude is reported in Nielsen data for Chile by Labbé (2013). This effect is also present in the French CPI, which may also include some small price changes due to substitutions. For the French CPI excluding (log percent) price changes for which $|\Delta p| \leq 1$ as opposed to only excluding those price changes for which $|\Delta p| \leq 1/10$, reduces kurtosis from 7.21 to 6.33 (see lines 4 vs 11 of Table 7). For the Norvegian CPI Wulfsberg (2010) finds that kurtosis is also sensitive to both large and small price changes, removing the 1 and 99 percentiles decreases kurtosis from 8.1 to 5.7.

2.3 A comparison with the US data

To assess whether the patterns documented above are specific to France we compare our data with the US figures presented, respectively, in Klenow and Kryvtsov (2008) and in Eichenbaum et al. (2013). Figure 2 plots four histograms: two are price changes from the US and France, while the other two are theoretical benchmarks. The first one (in red) is the distribution of standardized (weighted) price changes (excluding sales) for the US based on Figure 3 of Klenow and Kryvtsov (2008).[17] Since the distribution is truncated at -3 and +3, its standard deviation is 0.83 instead of 1, its kurtosis is 6.95. The second histogram (in blue) is the distribution of the standardized price changes (excl. sales) for the French CPI, constructed using the trimming criteria used for the US. This distribution has a standard deviation 0.95 and a kurtosis of 4.42. The smaller standard deviation and much smaller

[16] In the French CPI we have observed cases where a single large price change which reverts to its original value, which corresponds to consecutive prices that are equal except for the transposition of a digit, which is almost surely a clerical error.

[17] The histogram has twenty four bins, spaced every 0.25 units, of the distribution of standardized regular price changes (excl. sales). The standardization was done by ELI, the narrowest categories of goods. After standardization the distributions are weighed according to the CPI weight.

kurtosis than in Table 1 are due to the discretization and truncation. To see the effect of these treatment of the data, note that Klenow and Malin (2010) report a kurtosis of 10 for posted prices and 17.4 for regular prices, without discretizing, censoring, or standardizing the data. For comparison Vavra (2013) finds that after trimming the data in a way similar to our treatment of the French data, but without standardizing it, the kurtosis of US CPI price changes is 6.4. On the other hand, Vavra finds that the kurtosis of standardized price changes is 4.9.[18] The figure also reports the standardized Normal and Laplace distributions (discretized and truncated). The main outcome of Figure 2 is that the histogram of standardized, non-sales, price change are very similar in France and the US. Furthermore, in both cases the shape is closer to that of a Laplace distribution than to a Gaussian one (and consistently with previous sub-section, in both cases we conjecture measurement error explains why these distribution are actually more peaked than the Laplace).

Table 3: Fraction of small price changes: US and French CPI

Moments for the absolute value of price changes: $\|\Delta p\|$	France	US	Normal	Laplace
Average $\|\Delta p\|$	9.2	14.0		
Fraction of $\|\Delta p\|$ below 1%	11.8	12.5		
Fraction of $\|\Delta p\|$ below 2.5%	32.5	24.0		
Fraction of $\|\Delta p\|$ below 5%	57.1	40.6		
Fraction of $\|\Delta p\|$ below $(1/14) \cdot \mathbb{E}(\|\Delta p\|)$	2.4	12.5	4.5	6.9
Fraction of $\|\Delta p\|$ below $(2.5/14) \cdot \mathbb{E}(\|\Delta p\|)$	13.5	24.0	11.3	16.4
Fraction of $\|\Delta p\|$ below $(5/14) \cdot \mathbb{E}(\|\Delta p\|)$	28.7	40.6	22.4	30.0
Number of obs	1,542,586	1,047,547		

For France the source is INSEE monthly price records from the French CPI (2003:4 to 2011:4). Coverage is around 65% of CPI weight since rents, and prices of fresh food and centrally collected items (e.g. electricity, train and airplane tickets) are not included in the dataset. Frequency of price change is monthly, in percent. Size of price change are the first-difference in the logarithm of price per unit, expressed in percent. Data are trimmed as in the baseline of Table 1. Observations with imputed prices or quality change are discarded. Moments are computed aggregating all prices changes using CPI weights at the product level. The US data are taken from Eichenbaum et al. (2013) Table 1, and refer to "Posted price changes" from 1998:1 to 2011:6. The mean absolute size of price changes is taken from Klenow and Kryvtsov (2008) table III where data are from 1998:1 to 2005:1. Figures for the US are weighted and cover around 70% of the CPI (US CPI includes owners equivalent rents, while French CPI does not). In the third panel we compute the threshold for defining small price changes as fraction of the mean so as to match the US figures in column 2 of the second panel. The Normal and Laplace distributions used in the last two columns have a zero mean and, without loss of generality, standard deviation equal to one.

Table 3 uses the same thresholds of Eichenbaum et al. (2013) to measure the fraction

[18] See Table IV and footnote 6 of Vavra (2013) for the specifics on the trimming. We thank Vavra for providing this statistics which is not available in his paper.

of small price changes. The presence of small price changes (in absolute value) is at first sight a more prominent fact in France than in the US. One factor that may contribute to explaining this pattern is the fact that sales are less prevalent in France. Measurement error, as discussed above, may play a role. We nevertheless observe that, if we define small price change as relative to the mean average price change, rather than with an absolute threshold, the fraction of small price change appears to be lower in France than in the US, as shown in Table 3.

Table 4 provides a further comparison based on datasets presumably less subject to measurement errors. For France we use data from the BPP, and those from hypermarkets in the CPI dataset. For the US we use the results on scanner data reported by Midrigan (2011), as well as results on scanner data from a large US supermarket chain. As reflected by our three summary statistics, the distribution is somewhat more peaked in France; for instance the kurtosis is 5 in the BPP against 3.5 in Midrigan (2011). However, these results still support the notion that the share of small price changes is sizable in both countries.

Overall we conclude that, after accounting for heterogeneity and extrapolating our estimate of the effect of measurement error from large retailers to the rest of vendors, the prevalence of both small and large price changes appears relevant in France as well as in the US. The shape of standardized empirical distribution of price changes lays in between a Normal and a Laplace distribution even though the distribution appears close to a Normal in the US and closer to Laplace in France. As a benchmark we summarize the shapes of the standardized distributions for broad price indices, after taking measurement error into account, as having kurtosis of about 4 for US and of about 5 for France.

Table 4: Comparison across datasets for large Hypermarkets in France and the US

	France		**US**	
	CPI	BPP data	scanner data	Midrigan (2009)
	Statistics for standardized price changes: z			
mean of $\|z\|$	0.65	0.70	0.80	-
% below 0.50 mean $\|z\|$	45	39	31	29
% below 0.25 mean $\|z\|$	24	21	20	13
kurtosis of z	10	5	2.7	3.5

All price changes including sales. The BPP statistics for France are an average of the ones reported in Table 2. The US scanner data in the third column are from a large US supermarket chain for which individual observations are average weekly prices. For this data set we remove the observations for which percentage log prices Δp satisfy $|\Delta p| \geq 100 \log(10/3)$ or $|\Delta p| \leq 1/10$. The data from Midrigan (2009) are taken from his Table 1 and 2b, using simple averages of the AC Nielsen and Dominick's scanner data. Midrigan standardizes price changes by dividing by the mean absolute value of the price changes at the product-store-month level. He also removes price changes smaller than 1 cent or larger than 100%.

3 A tractable menu cost model

This section presents a menu cost model aimed at qualitatively matching some of the patterns documented above. In the canonical menu cost model price adjustments occur when a threshold is hit, so that the implied distribution of price changes fails to generate the small changes that appear in the data (see the discussion in Midrigan (2011); Cavallo (2010); Alvarez and Lippi (2013b)). The model that we propose here is able to produce a large mass of small price changes and the positive excess kurtosis that we documented above. Two ingredients are key to this end: (i) the random menu costs and that (ii) the menu cost faced by the firm, ψ, applies to a bundle of n goods, so that after paying the fixed cost the firm can reprice one or all goods at no extra cost. Each of these assumptions individually is capable to generate some small price changes and higher kurtosis than in a canonical model where $n = 1$ and where menu costs are constant. The assumption of random menu costs is key to generate a positive excess kurtosis in the distribution of price changes. The combination of the two however is important: in the models where $n = 1$ (with or without random menu costs) the distribution of price changes has a mass point at the adjustment threshold, a feature that is in stark contrast with the evidence discussed above. The prominence of "large" price changes (i.e. a "U shaped" distribution) persists even in a model with $n = 2$, as in Midrigan (2011) where the distribution of price changes asymptotes near the adjustment threshold, or $n = 3$. We show below that in order to generate a shape of the size distribution that is comparable to the one in the data one needs $n \geq 6$.

Our model relates to the seminal work by Dotsey, King, and Wolman (1999) on the aggregate effects in models where firms face a menu cost for prices changes that is random and is drawn from a distribution with a smooth density. In our model the menu costs is either ψ with probability $1 - \lambda dt$ (in a time period of length dt) or 0 with probability λdt. Additionally, our model has persistent idiosyncratic shocks to marginal cost – as well as multi-product firms– which combined with the random menu cost create a rich class of distributions for price changes (including Binomial, Normal and Laplace), where in Dotsey, King, and Wolman (1999) random menu cost are the only source of idiosyncratic variation in prices.[19] Other researchers find the models with both random menu cost and idiosyncratic shock a useful, such Dotsey, King, and Wolman (2009) who extended a version of their model, Nakamura and Steinsson (2010), whose "CalvoPlus" model has also two values for the menu costs and which in general is the closest to our model with $n = 1$, and more recently the work by Vavra (2013).

For ease of exposition we first illustrate the model with random menu costs where the

[19]There are other differences, such as the absence of capital accumulation in our set-up, and small difference in functional forms for money demand and preferences.

firm sells a single good (i.e. $n = 1$) and then extend the model to include any number of goods $n > 1$.

3.1 A random menu cost problem for a firm selling $n = 1$ good.

Consider a firm whose profit-maximizing price at time t, $p^*(t)$, follows the process $\mathrm{d}p^*(t) = \sigma\,\mathrm{d}W(t)$ where $W(t)$ is a standard brownian motion with no drift and i.i.d. innovations with standard deviation σ. The technology to change prices is as follows: to change the price at will the firm needs to incur a fixed menu cost of size ψ. However, with some probability the firm receives an opportunity to adjust the price "for free". Assume this probability is Poisson, i.e. that the free-adjustments have a constant hazard rate per unit of time, equal to λ. Let $p(t)$ denote the "price gap" at time t, i.e. the difference between the actual sale price $P(t)$ and the profit maximizing price $p^*(t)$, i.e. $p(t) \equiv P(t) - p^*(t)$. The instantaneous firm losses (i.e. reduction in profits) created by the price gap are given by the quadratic: $B\,p^2(t)$. Let $v(p)$ be the present-value cost function for a firm with price gap p. Upon the arrival of a free adjustment opportunity the firm optimally resets the price gap to zero, hence the Bellman equation for the range of inaction reads:

$$r\;v(p) = Bp^2 + \lambda\left[v(0) - v(p)\right] + \frac{\sigma^2}{2}\;v''(p), \qquad \text{for } p \in (0, \bar{p})\;,$$

where $\bar{p}$ is the boundary of the region in which inaction is optimal. This equation states that the flow value of the Bellman equation is given by the instantaneous losses, Bp^2, plus the expected change in the value function, which is due either to a free adjustment (with rate λ in which case the price gap is reset to zero) or to the volatility of shocks σ^2 (there is no first order derivative of the value function since the price gaps have no drift).[20] The value matching and smooth pasting conditions are given by $v(\bar{p}) = v(0) + \psi$ and $v'(\bar{p}) = 0$.

Next we describe the optimal decision rules and some key statistics implied by the model with $n = 1$ (see Appendix D for the derivation). A Taylor expansion of the value function yields the following approximate optimal threshold $\bar{p} = \left(\frac{6\psi\sigma^2}{B}\right)^{\frac{1}{4}}$ which is accurate when ψ/B is small.[21] We comment on two properties of the decision rule of this problem which are proved later for the more general case: the value function, and the optimal decision rules, are a function of $\lambda + r$, as opposed to each of them separately. Intuitively this is because when a free adjustment opportunity occurs the price gap is adjusted, so that λ acts as an addition to the discount factor. Second, for a small value of ψ/B or a small value of $\lambda + r$,

[20]See Dixit (1991) and Stokey (2008) for more discussions of continuous time Bellman equations.

[21] Exactly the same expression was established by Barro (1972); Dixit (1991) for the case in which $\lambda = 0$. Below we discuss an approximate threshold for the case in which ψ is large.

the value of $\bar{p}$ is insensitive to $\lambda + r$, as the previous approximation shows. More precisely, the derivative of $\bar{p}$ with respect to $\lambda + r$ is zero as ψ/B or $\lambda + r$ tend to zero. This property, which was known for the case of $\lambda = 0$, extends to the case where $\lambda + r > 0$ using the first property of the decision rule.

Computing the expected time between adjustments yields an expression for the average number of adjustments per period, N_a, which we use to write an expression for the fraction of free adjustments over the total number of adjustments, ℓ, as

$$\ell \equiv \frac{\lambda}{N_a} = \frac{e^{\sqrt{2\phi}} + e^{-\sqrt{2\phi}} - 2}{e^{\sqrt{2\phi}} + e^{-\sqrt{2\phi}}} \in (0,1) \qquad \text{where we define} \qquad \phi \equiv \frac{\lambda \bar{p}^2}{\sigma^2}$$

which shows that the fraction of free adjustments ℓ depends only on the parameter ϕ. The parameter ϕ can be interpreted as the ratio between λ, the number of free adjustments, and $\sigma^2/\bar{p}^2$, the number of adjustments in a model where $\lambda = 0$ and the threshold policy $\bar{p}$ is followed.

The distribution of price changes $w(\Delta p_i)$ is symmetric around $\Delta p_i = 0$. This distribution has a mass point at $\Delta p_i = \pm\bar{p}$ with probability $1 - \ell$, i.e. this is the fraction of price changes that occurs because the price gap reaches the boundaries of the inaction region. The remaining fraction of price changes, ℓ occurs when a free adjustment opportunity arrives, at which time the price gap is set to zero. Price changes in the range $p \in (-\bar{p}, \bar{p})$ have a density $\ell\ h(p)$ where $h(p)$ denotes the density of the invariant distribution of price gaps

$$h(p) = \frac{\sqrt{2\phi}}{2\bar{p}\left(e^{\sqrt{2\phi}} - 1\right)^2}\left(e^{\sqrt{2\phi}\left(2 - \frac{|p|}{\bar{p}}\right)} - e^{\sqrt{2\phi}\frac{|p|}{\bar{p}}}\right) \quad \text{for} \quad p \in [-\bar{p}, \bar{p}]\ .$$

Thus the distribution of price changes is given by

$$\begin{cases} \Pr\left(\Delta p_i = -\bar{p}\right) = \Pr\left(\Delta p_i = \bar{p}\right) & = \frac{1}{2}\left(1 - \ell\right) \\ \Pr\left(\Delta p_i \in\ dp\right) & = \ell\ h(p)dp \equiv w(\Delta p_i)dp \qquad \text{for } p \in (-\bar{p}, \bar{p}) \end{cases}$$

which is a symmetric "tent shaped" distribution in the $(-\bar{p}, \bar{p})$ interval with the two mass points at the boundaries $\pm\bar{p}$. As detailed below the kurtosis of this distribution is increasing in λ, and in particular the distribution of price changes is more peaked than that of a standard menu cost model $\lambda = 0$.

We make two remarks about this simple model which will hold, and be generalized, in the more general model developed next. The first one is that the shape of the distribution of price changes depends *only* on the fraction of free adjustments ℓ (or, equivalently, on ϕ). This means that two economies, or sectors, that differ in the standard deviation of price changes $Std(\Delta p_i)$ and/or in the frequency of price adjustment N_a will display a distribution of price changes with exactly the same shape (once its scale is adjusted) provided that they have the same value of ℓ. This property is useful to aggregate the sectors of an economy that are heterogenous in their steady state features (N_a, $Std(\Delta p_i)$). Because of this property the ratio of moments from the size distribution of price changes, such as kurtosis, are scale free and can be used to retrieve information on ϕ. The second property, which we state here and prove below for the more general economy, is that the "shape" of the impulse response function of this economy to a (once and for all) monetary shock depends *only* on ℓ. We will show how one can simply scale (or relabel) one or both axes of an impulse function to analyze economies with the same ℓ that differ in either N_a or $Std(\Delta p_i)$.

3.2 Extending the model to multi-product firms

This section incorporates the model with free adjustment opportunities discussed above into the model of Alvarez and Lippi (2013b) where the firm is selling n goods, as opposed to a single good, but pays a single fixed adjustment cost to change the n prices. We incorporate this feature for several reasons. First, as explained above, in the model with $n = 1$ good there is a mass point on price changes of size $|\Delta p_i| = \bar{p}$. There is no evidence of this in any data set we can find. Second, and related to the previous point, in the model with $n = 1$ a simple estimate of $\bar{p}$ will be the highest price change. We propose to use a different one, since this order statistic is both difficult to measure in practice and its role to measure $\bar{p}$ is very sensitive to the specification of the model. Third, the model with $\lambda = 0$ has a kurtosis that increases with n, hence providing and alternative to randomness on fixed cost, as discussed below. Fourth, for large n and $\lambda = 0$ the distribution of price changes tends to the Normal distribution, which is both a nice benchmark and an accurate description of the price changes for some sectors. Fifth, the multi-product model with (i.e. $n > 1$) has an alternative, broader, interpretation for the menu cost ψ. In this case one can assume that the firm freely observes the profits for all products, but not the individual ones, unless it either pays the cost ψ or a free observation opportunity arrives, in which case is able to set the optimal price to each of them. This is useful, because it allows a broader interpretation of menu cost, including not only the physical cost of changes prices but also those related to gathering the information for individual products.[22]

We now briefly describe the setup of the firm problem with n products. As before the

[22] As an example, see Chakrabarti and Scholnick (2007) who argue that for stores as Amazon or Barnes and Noble physical menu cost are small, yet prices change infrequently, and thus conclude that the cost may be of a different nature. Interestingly, they find that for such retailers price changes are synchronized across products, which is an implication of the multi-product model.

free adjustment opportunities are independent of the driving processes $\{W_i(t)\}$ for price gaps, and arrive according to a Poisson process with constant intensity λ. In between price adjustments each of the price gaps evolves according to a Brownian motion $\mathrm{d}p_i(t) = \sigma \, \mathrm{d}W_i(t)$. It is assumed that all price gaps are subject to the same variance σ^2 and that the innovations are independent across price gaps.[23]

We assume that, when the opportunity arrives, the firm can adjust all prices without paying the cost ψ. The analysis of the multi product problem can be greatly simplified by using the sum of the squared price gaps, $y \equiv ||p||^2$, as a state, as done in Alvarez and Lippi (2013b). The scalar y summarizes the state because the period objective function can be written as a function of it and because, from an application of Ito's lemma, one can derive one dimensional diffusion which describes its behavior, namely

$$\mathrm{d}y = n\sigma^2 \, \mathrm{d}t \, + \, 2\,\sigma\sqrt{y}\,\mathrm{d}W$$

where W is a standard Brownian motion.

Using N_a and $Var(\Delta p_i)$ to denote the frequency and the (cross sectional) variance of the price changes of product i the next proposition establishes a useful relationship that holds in a large class of models for *any policy* for price changes, which we describe by a stopping time rule:

Proposition 1 *Let τ describe the time at which a price change takes place, so that all price gaps are closed. Assume the stopping time treats each of the n price gaps symmetrically. For any finite stopping time τ we have:*

$$N_a \cdot Var(\Delta p_i) = \sigma^2 \; . \tag{2}$$

The proposition highlights the trade-off for the firm's policy: more frequent adjustments are required to have smaller price gaps. We underline that equation (2) holds for any stopping rule, not just for the optimal one. See Appendix A for the proof, where the reader can verify that the key assumptions are the random walks and symmetry; indeed equation (2) holds for a larger class of models, for instance those with correlated price gaps and a richer class of random adjustment cost.

Upon the arrival of a free adjustment opportunity the firm will set the price gap to zero,

[23] Alvarez and Lippi (2013b) discuss the case with correlated price gaps. Intuitively, as correlation increase the model becomes more similar to the $n = 1$ case, since the price changes of a firm become more similar.

hence the Bellman equation for the range of inaction reads:[24]

$$r\ v(y) = By + \lambda\left[v(0) - v(y)\right] + n\sigma^2\ v'(y) + 2\sigma^2 y\ v''(y), \quad \text{ for } y \in (0, \bar{y})\ , \tag{3}$$

where By is the sum of the deviation from the optimal profits from the n goods. The use of the one dimensional y instead of the vector $(p_1, ..., p_n)$ simplifies the problem substantially.

We note that given the symmetry of the problem after an adjustment of the n prices the firm will set each of the price gap to zero, i.e. will set $||p||^2 = y = 0$. The value matching condition is then $v(0) + \psi = v(\bar{y})$, which uses that when y reaches a critical value, denoted by $\bar{y}$, by paying the fixed cost ψ the firm can change the n prices. The smooth pasting condition is $v'(\bar{y}) = 0$.

The next lemma establishes how to solve for $\bar{y}$ using the solution of a simpler problem where $\lambda = 0$ discussed in Alvarez and Lippi (2013b). It turns out that a simple change of variables allows us to use the solution for the case of $\lambda = 0$ with the solution for the case of interest in this paper. The change of variables consists on using $r + \lambda$ as the interest rate in the solution of the problem with $\lambda = 0$. We have:

Lemma 1 *Let $\bar{y}(r, \lambda)$ and $v(y; r, \lambda)$ be the optimal value function and adjustment threshold for a problem with discount rate r and arrival rate λ. Then $v(y; r, \lambda) = v(y; r + \lambda, 0) + \frac{\lambda}{r} v(0; r + \lambda, 0)$ for all $y \geq 0$ and thus $\bar{y}(r, \lambda) = \bar{y}(r + \lambda, 0)$.*

The proof of this lemma follows immediately from a guess and verify strategy. The lemma allows us to use the characterization of $\bar{y}$ with respect to r given in Proposition 4 of Alvarez and Lippi (2013b) to study the effect of $r + \lambda$ on $\bar{y}$. In Appendix E we write the analytical solution for the value function, and give more details on it. The next proposition summarizes that result and extends the characterization of the optimal threshold to the case where ψ is large, a case that is useful to understand the behavior of an economy with a lot of free adjustments opportunity as in a Calvo mechanism (see Appendix A for the proof).

Proposition 2 *Assume $\sigma^2 > 0$, $n \geq 1$, $\lambda + r > 0$ and $B > 0$, and let $\bar{y}$ be the threshold for the optimal decision rule. We then have that:*

1. *As $\psi \to 0$ then $\frac{\bar{y}}{\sqrt{2(n+2)\sigma^2 \frac{\psi}{B}}} \to 1$ or $\bar{y} \approx \sqrt{2(n+2)\sigma^2 \frac{\psi}{B}}$.*
2. *As $\psi \to \infty$ we have $\frac{\bar{y}}{\psi} \to (r + \lambda)/B$ or $\bar{y} \approx \frac{\psi}{B}(r + \lambda)$. Moreover this also holds for large n and large $\frac{\psi}{n}$, namely $\lim_{\psi/n \to \infty} \lim_{n \to \infty} \frac{\bar{y}/n}{\psi/n} = (r + \lambda)/B$ or $\frac{\bar{y}}{n} \approx \frac{\psi/n}{B}(r + \lambda)$.*

[24] With a slight abuse of notation we use $v(\cdot)$ to denote the value function defined over the sum of the squared price gaps y, while in the case of $n = 1$ the value function was defined over the price gap p.

The proposition shows that $\bar{y}$ is approximately constant with respect to λ for small values of ψ, so that for small menu costs the result is the well known quartic root formula (recall that y has the units of a squared price gap) and the inaction region is increasing in the variance of the shock, due to the higher option value. Interestingly, and novel in the literature, the second part of the proposition shows that for large values of the adjustment cost the rule becomes a square root and that the optimal threshold does not depends on σ, which shows that for large adjustment costs the option value component of the decision becomes negligible. Moreover, when the menu costs are large the threshold $\bar{y}$ is increasing in λ: the prospect of receiving a free adjustment tomorrow increases inaction today.

We now turn to the discussion of the model implications for the frequency of price changes. We let $N_a(\bar{y};\lambda)$ be the expected number of adjustments per unit of time of a model with a given λ and $\bar{y}$. We establish the following (see Appendix A for the proof):

Proposition 3 *The fraction of free adjustments is $\ell = \lambda/N_a = \mathcal{L}(\phi, n)$, where*

$$\mathcal{L}(\phi,n) \equiv \frac{\phi\left[1+\sum_{i=1}^{\infty}\left(\prod_{k=1}^{i}\frac{n}{(k+1)\ (n+2k)}\right)\phi^i\right]}{1+\phi\left[1+\sum_{i=1}^{\infty}\left(\prod_{k=1}^{i}\frac{n}{(k+1)\ (n+2k)}\right)\phi^i\right]} \quad \textit{where} \quad \phi \equiv \frac{\lambda\bar{y}}{n\sigma^2} \tag{4}$$

The proposition shows that ℓ, a key parameter for the behavior of the model, is a function only of two variables: n and ϕ. As for the $n=1$ model, the parameter ϕ can be interpreted as the ratio between λ, the number of free adjustments, and $n\sigma^2/\bar{p}^2$, the number of adjustments in a model where $\lambda = 0$ and the threshold policy $\bar{y}$ is followed. A second order approximation of $\mathcal{L}(\phi,n)$ shows that λ has a negligible effect on the frequency of adjustment N_a when $\bar{y}$ is small, i.e. the first order term is the same as the one for the model with $\lambda = 0$.[25]

We now turn to characterizing the invariant distribution of y for the case where $\lambda > 0$, a key ingredient to compute the size-distribution of price changes. The density of the invariant distribution solves the Kolmogorov forward equation: $\frac{\lambda}{2\sigma^2}f(y) = f''(y)y - \left(\frac{n}{2}-2\right)f'(y)$ for $y \in (0,\bar{y})$, with the two boundary conditions $f(\bar{y}) = 0$ and $\int_0^{\bar{y}} f(y)dy = 1$. It is clear from these conditions that $f(\cdot)$ is uniquely defined for a given triplet: $\bar{y} > 0$, $n \geq 1$ and $\lambda/\sigma^2 \geq 0$. The general solution of this ODE is

$$f(y) = \left(\frac{\lambda y}{2\sigma^2}\right)^{\left(\frac{n}{4}-\frac{1}{2}\right)}\left[C_1\ I_\nu\left(2\sqrt{\frac{\lambda y}{2\sigma^2}}\right) + C_2\ K_\nu\left(2\sqrt{\frac{\lambda y}{2\sigma^2}}\right)\right] \tag{5}$$

where I_ν and K_ν are the modified Bessel functions of the first and second kind, C_1, C_2 are two arbitrary constants and $\nu = |\frac{n}{2}-1|$, see Zaitsev and Polyanin (2003) for a proof. The

[25]The expansion gives $\frac{1}{N_a} = \frac{\bar{y}}{n\sigma^2}\left[1-\lambda\frac{\bar{y}}{n\sigma^2}\frac{(n+4)}{(2n+2)}\right] + o\left(\lambda\left(\frac{\bar{y}}{\sigma^2}\right)^2\right)$ which shows that $1/N_a = \bar{y}/(n\sigma^2) + o(\bar{y})$.

constants C_1, C_2 are chosen to satisfy the two boundary conditions.[26] While the density in equation (5) depends on 3 constants n, ϕ and $\bar{y}$, its shape depends only on 2 constants, namely n and ϕ, as formally stated in Lemma 2 in Appendix A. The lemma shows that one can normalize $\bar{y}$ to 1 and compute the density for the corresponding ϕ.

We denote the marginal distribution of price changes by $w(\Delta p_i)$. Recall that firms change prices either when y first reaches $\bar{y}$ or when they get a free adjustment opportunity even though $y < \bar{y}$. Thus to construct the distribution of price changes we need three objects: the fraction of free adjustments ℓ, the invariant distribution $f(y)$ and the marginal distribution of price changes *conditional on a value of* y, $\omega(\Delta p_i; y)$ which, following Proposition 6 of Alvarez and Lippi (2013b) when $n \geq 2$, is

$$\omega(\Delta p_i; y) \;=\; \begin{cases} \frac{1}{Beta\left(\frac{n-1}{2},\frac{1}{2}\right)\,\sqrt{y}} \left(1 - \left(\frac{\Delta p_i}{\sqrt{y}}\right)^2\right)^{(n-3)/2} & \text{if} \quad (\Delta p_i)^2 \leq y \\ 0 & \text{if} \quad (\Delta p_i)^2 > y \end{cases} \tag{6}$$

where $Beta(\cdot,\cdot)$ denotes the Beta function. In this case the (cross-sectional) standard deviation of the price changes is $Std(\,\Delta p_i\,; y) = \sqrt{y/n}$. The marginal distribution of price changes $w(\Delta p_i)$ is given by

$$w(\Delta p_i) = \omega(\Delta p_i; \bar{y})\,(1-\ell) + \left[\int_0^{\bar{y}} \omega(\Delta p_i; y) f(y) dy\right] \ell \qquad \text{for } n \geq 2 \quad . \tag{7}$$

For the case when $n = 2$ the density of the price changes diverges at the boundaries of the domain where $\Delta p_i = \pm\sqrt{\bar{y}/n}$, as can be seen in Figure 3. This feature echoes the two mass points that occur in the $n = 1$ case where a non-zero mass of price changes occurs exactly at the boundaries. For $n \geq 6$ the shape of the density takes a tent-shape, similar to the one that is seen in the data. As the fraction of free adjustments approaches 1 the shape of the density function converges to the shape of the Laplace distribution. The next proposition shows that n and ℓ completely determine the shape of the distribution of price changes (see Appendix A for the proof):

Proposition 4 *Let $w(\Delta p_i; n, \ell, 1)$ be the density function for the price changes Δp_i in an economy with n goods, a share ℓ of free adjustments, and a unit standard deviation of price changes $Std(\Delta p_i) = 1$. This density function is homogenous of degree -1 in Δp_i and $Std(\Delta p_i)$, which implies*

$$w\left(S\,\Delta p_i; n, \ell, S\right) = \frac{1}{S} \;\; w\left(\Delta p_i; n, \ell, 1\right) \qquad \textit{for all} \quad S > 0. \tag{8}$$

[26] We note that both modified Bessel functions are positive, that $I_\nu(y)$ is exponentially increasing with $I_\nu(0) \geq 0$, and that $K_\nu(y)$ is exponentially decreasing with $K_\nu(0) = +\infty$.

The proposition establishes that the "shape" of the size distribution of price changes has 2 parameters: n and ℓ. Every two economies sharing these parameters will have the same size distribution of price changes once the scale is adjusted. The proposition implies that we can aggregate firms or industries that are heterogenous in terms of frequency N_a and standard deviation of price changes $Std(\Delta p_i)$ provided that n and ℓ are the same. Notice in particular that the frequency of price changes N_a does not have an independent effect on the distribution of price changes as long as ℓ remains constant.

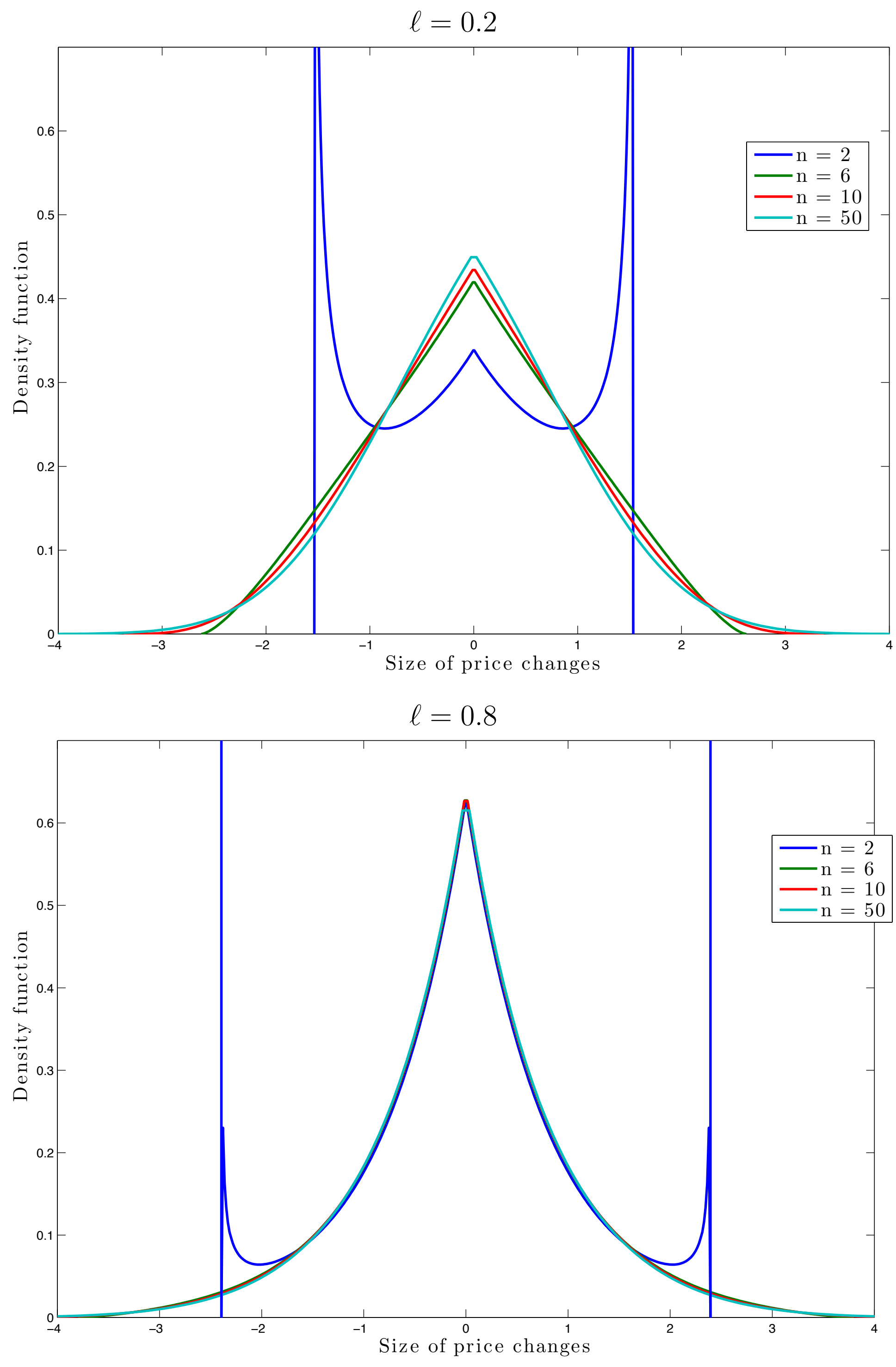
Notice that the distribution $w(\Delta p_i)$ is a mixture of the $\omega(\Delta p_i, y)$ densities. These densities are scaled versions of each other with different standard deviations. This increases the kurtosis of the distribution of price changes compared to the case where $\lambda = 0$. In particular Proposition 6 in Alvarez and Lippi (2013b) shows that the variance and kurtosis of $\omega(\Delta p_i, y)$ are given by y/n and $3n/(n+2)$ respectively. Using that Δp_i is distributed as a mixture of the $\omega(\Delta p_i, y)$, we can compute several moments

$$\begin{aligned}
\mathbb{E}(|\Delta p_i|) &= \frac{(1-\ell)\sqrt{\bar{y}} + \ell \int_0^{\bar{y}} \sqrt{y}\ f(y)dy}{\frac{n-1}{2}\ Beta\left(\frac{n-1}{2}, \frac{1}{2}\right)} \\
Var(\Delta p_i) &= (1-\ell)\frac{\bar{y}}{n} + \ell \int_0^{\bar{y}} \frac{y}{n}\ f(y)dy \\
kurt(\Delta p_i) &= \frac{3n}{2+n}\ \frac{(1-\ell)\,\bar{y}^2 + \ell \int_0^{\bar{y}} y^2\ f(y)dy}{\left[(1-\ell)\,\bar{y} + \ell \int_0^{\bar{y}} y\ f(y)dy\right]^2} > \frac{3n}{2+n}
\end{aligned} \tag{9}$$

It is immediate from Proposition 4 that the value of the kurtosis and the value of the ratio $\mathbb{E}(|\Delta p_i|)/Std(\Delta p_i)$ depend only on two parameters: n and ℓ. For instance, if one were to change the parameters ψ/B, λ and σ^2 keeping the same values for ℓ and n, the kurtosis of the price changes will be the same. The inequality that appears in the third line is a well known result: the mixture of distributions with the same kurtosis but with different variances has higher kurtosis, which itself follows from Jensen's inequality. Moreover as $\bar{y} \to \infty$ (as it will happen if $\psi/B \to \infty$) then $\ell \to 1$ and one can show that $kurt(\Delta p_i) \to 6$. This is because as $\bar{y} \to \infty$ the price changes in each coordinate are independent, and hence it has the same distribution than in the case of $n = 1$, i.e. a Laplace distribution. The maximum kurtosis that the model with free adjustments can produce is 6, which happens in the limiting case in which *all* adjustments are free (e.g. when $\ell \to 1$ and $\bar{y} \to \infty$) and is independent of the number of products that are priced by the firm, n. Table 5 computes the kurtosis of the model for the intermediate cases in which only a fraction of adjustments are free.[27] The columns correspond to different values of n, the number of goods. Each line corresponds to

[27]More statistics, concerning e.g. the fraction of small price changes, are shown in Appendix I.

Figure 3: Size distribution of price changes



Note: All distributions are zero mean with unit standard deviation. As stated in Proposition 4 the shape of this distribution only depends on ℓ and n.

a different proportion of free adjustments: ℓ. When the fraction of free adjustment is small (first and second line of the table) the model behaves essentially like the one described in Alvarez and Lippi (2013b): kurtosis is increasing in n up to a level of about 3. For instance, if $\ell = 0$ and $n \to \infty$, the kurtosis converges to 3 since the distribution of price changes at the time of adjusting for each firm becomes normal; this value is the highest that our *purely* multi-product firms can produce. For any n, as the fraction of free adjustments ℓ increases the kurtosis increases towards 6, and becomes less responsive to n.[28] Summarizing, both the random menu cost and the multi product assumption (for $n > 2$) produce more frequent small and large price changes, as can be seen in the densities of Figure 3. This is readily translated into higher values of kurtosis, as can be seen in Table 5.

Table 5: Model statistic for the kurtosis of Price changes

% of free adjustments:	number of products n					
ℓ	1	2	4	6	10	50
0 %	1.0	1.5	2.0	2.3	2.5	2.9
10%	1.1	1.6	2.1	2.4	2.6	3.0
20%	1.2	1.7	2.2	2.5	2.7	3.1
50%	1.6	2.2	2.7	3.0	3.2	3.6
70%	2.1	2.8	3.3	3.5	3.7	4.1
80%	2.6	3.2	3.7	3.9	4.1	4.4
90%	3.4	3.9	4.3	4.5	4.7	4.9
95%	4.1	4.5	4.8	5.0	5.1	5.3
100%	6.0	6.0	6.0	6.0	6.0	6.0

To conclude the description of the model and its predicted moments, we summarize some special cases nested by our setup. The Golosov-Lucas model is obtained when $n = 1$ and $\ell = 0$, implying a kurtosis of 1. The Taylor, or equivalently "rational inattentiveness", model is obtained when $n = \infty$ and $\ell = 0$, implying a kurtosis of 3. Our version of the Calvo model is obtained for $\ell \to 1$, for all values of n, implying a kurtosis of 6. The special cases of 2 well known models can be obtained: the "CalvoPlus" model by Nakamura and Steinsson (2010) for the special case of no intermediate goods ($s_m = 0$ using their notation), is obtained assuming $n = 1$ and $\ell \in (0, 1)$. The multiproduct model of Midrigan (2011) can be proxied assuming $n = 2$ and modeling the fat tailed shock by assuming $\ell \in (0, 1)$.[29] Depending on

[28] The relationship between the fraction of free adjustment and the "mass of small price changes", a statistic that is closely related to kurtosis, was noticed in the numerical simulations of Nakamura and Steinsson (2010) (see their footnote 15, where their "frequency of low repricing opportunities", $1 - \alpha$, is essentially our ℓ).

[29] Yet it differs in that the fat-tailed shocks mainly contribute to large price changes, see Appendix F for a formal discussion.

the parameter choice for ℓ, the last two models can generate a kurtosis between 1 and 6. In fact, the parametrization of both models is such that the implied kurtosis is on the high side. It is shown in Section 4.2 that higher values of kurtosis are essential to explain why the real effects in those models are closer to Calvo than to Golosov-Lucas. It appears that the calibration choice of ℓ is important to determine the real effects of monetary policy. The next subsection discusses how the empirical evidence can be useful in guiding the calibration.

3.3 On the implied cost of price adjustment

In this section we give a thorough characterization of the implications of the model for the size of the menu cost, i.e. a mapping between observable statistics and the value of ψ. The model presented above has four independent parameters: the scaled menu cost ψ/B, the volatility of shocks σ, the number of goods n and the rate of free adjustment opportunities λ. We find it convenient to pin down two of these parameters by matching observable statistics that are available in micro datasets: the frequency and the variance of price changes: N_a and $Var(\Delta p_i)$. Given these statistics the model has two residual parameters: $\ell = \lambda/N_a$ and n. The parametrization of the model can thus be usefully interpreted as choosing these 2 parameters to match two empirical observations. It was shown in Proposition 4 how ℓ and n shaped the distribution of price changes, in particular its kurtosis. This section shows how ℓ and n map into the cost of price adjustments, for given values of N_a and $Var(\Delta p_i)$. This is useful because it shows how to discipline the model parametrization using evidence on the cost of price adjustment, a topic that has been explored by e.g. Levy et al. (1997); Zbaracki et al. (2004).

We consider two measures for the cost of price adjustment: the first one is the cost of a single price adjustment (per product) as a fraction of the (per product) profits: ψ/n. This is the cost that a firm must pay if it decides to adjust all prices instantaneously (i.e. without waiting for the possibility of a free adjustment). Measuring this cost as a fraction of profits transforms these magnitudes into units that have an intuitive interpretation. The second measure is the average flow cost of price adjustment given by: $N_a \, \frac{\psi}{n} (1-\ell)$. This cost measures the average amount of resources that the firm pays to adjust prices per period. The difference between the two measures should be clear: when all price adjustments are costly, as in a model where $\ell = 0$, the relevant measure of price adjustments is ψ/n, so that the total flow cost of price adjustment borne by a firm per year is $N_a \, \psi/n$. Allowing for a fraction of adjustments to be free the total flow cost must be multiplied by $1-\ell$, as some of the adjustments that occur during the period are free. The latter measure is useful because it relates more directly to what has been measured in the data by the empirical studies mentioned above, namely the "average" cost of a price adjustment. The next proposition analyzes the mapping between the scaled menu cost ψ/n , and $B, \ell, n,$ N_a and $Var(\Delta p_i)$.

Proposition 5 *Fix the number of products $n \geq 1$ and let $r \downarrow 0$. There is a unique triplet $(\sigma^2, \lambda, \psi)$ consistent with any triplet $\ell \in [0,1], Var(\Delta p_i) > 0$ and $N_a > 0$. Moreover, fixing any value ℓ, the menu cost $\psi \geq 0$ can be written as:*

$$\frac{\psi}{n} = B \; \frac{Var(\Delta p_i)}{N_a} \; \Psi\left(n \, , \, \ell\right) \tag{10}$$

where Ψ is only a function of (n, ℓ). For all $n \geq 1$ the function $\Psi(n, \cdot)$ satisfies:

$$\lim_{\ell \to 0} \Psi\left(n, \, \ell\right) = \frac{n}{2\left(n+2\right)} \; , \quad \lim_{\ell \to 1} \Psi\left(n, \, \ell\right) = \infty \; , \quad \lim_{\ell \to 1} \Psi\left(n, \, \ell\right)\left(1-\ell\right) = 0 \; , \tag{11}$$

$$\lim_{\ell \to 1} \frac{\Psi(n', \ell)/n'}{\Psi(n, \ell)/n} \leq 1 \;\; \text{for } n' \geq n, \;\; \text{and} \;\; \lim_{n \to \infty} \frac{\Psi(n, \ell)/n}{\Psi(1, \ell)/1} \to 0 \;\; \text{as } \ell \to 1 \, . \tag{12}$$

We comment and interpret the results of Proposition 5: first, equation (10) shows that for any fixed $n \geq 1$ and $\ell \in [0, 1]$ the menu cost ψ is proportional to the ratio $Var(\Delta p_i)/N_a$. This is very intuitive: everything else the same, economies with higher frequency of price changes are obtained by having a proportionally lower menu cost, and economies with more extreme price changes, are obtained with a proportionally higher menu cost. Second, equation (10) shows that the menu cost is proportional to B, which measures the benefits of closing a (unit square) price gap. The parameter B is related to the curvature of the profit function, and thus –as standard– it relates to demand elasticities and mark-ups.[30] Using a fully specified microeconomic problem where firms face a constant demand elasticity η (equal across products) gives that $B = \eta(\eta - 1)/2$, which can be written in terms of the markup over marginal costs $\mu \equiv 1/(\eta - 1)$ so that $B = (1 + \mu)/(2\mu^2)$.[31] The last expression is useful to calibrate the model using empirical estimates of the markup such as the ones by Christopoulou and Vermeulen (2012): the estimated markups average around 28% for the US manufacturing sector, and around 36% for market services (slightly smaller values are obtained for France, see their Table 1).[32] A similar value for the US, namely a markup rate of about 33%, is used by Nakamura and Steinsson (2010).

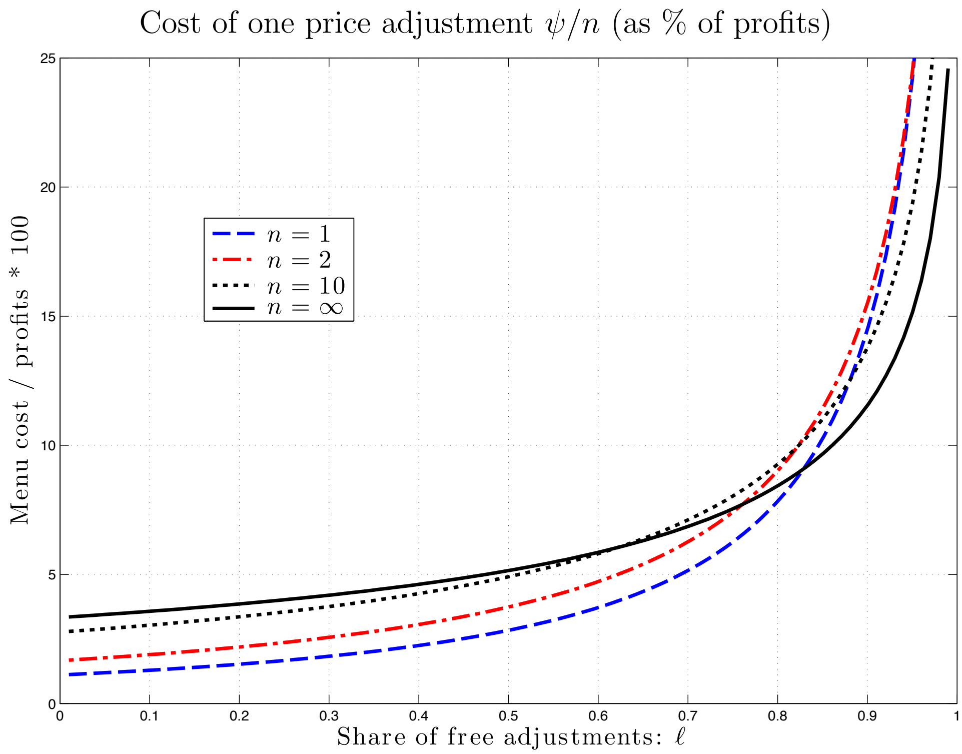
Next we comment on the effect of ℓ and n on the implied menu cost, fixing $B \, Var(\Delta p_i)/N_a$, i.e. we comment on the function $\Psi(n, \ell)$. This function can be readily computed using our characterization of the optimal threshold, Lemma 1, the value of N_a and ℓ from Proposition 3

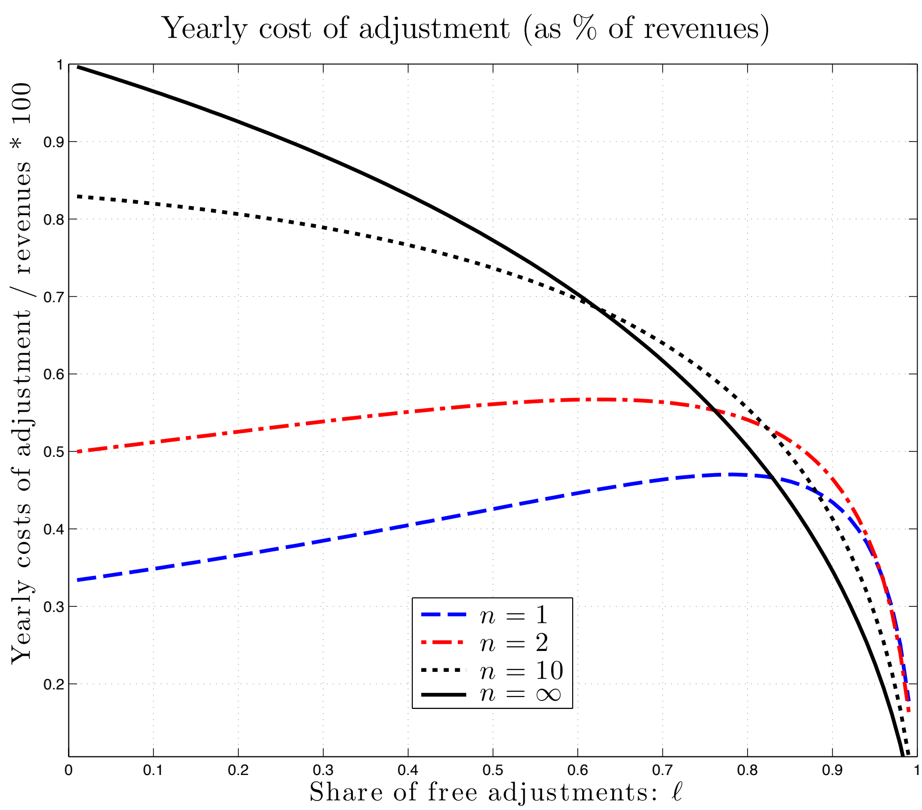
[30]See Appendix B in Alvarez and Lippi (2013b) for a derivation.

[31]Nakamura and Steinsson (2010) notice that lower markups (higher values of demand elasticity) η must imply higher menu costs, as shown by equation (10). Footnote 14 in their paper discusses evidence on the markup rates across several microeconomic studies and macro papers.

[32] The evidence for the US services is consistent with the gross margins, based on accounting data, reported in the Annual Retail Trade Survey by the US Census (see http://www.census.gov/retail/).

Figure 4: Implied cost of price adjustment





All economies in the figures feature $Std(\Delta p_i) = 0.10$ and a markup of 25%. For those in the top panel we set $N_a = 1.5$.

and the values of $N_a\, Var(\Delta p_i)$ from Proposition 1. Figure 4 plots the value of ψ/n for four values of n, and the following remark displays a closed form solution for the two limiting cases in which $n = 1$ and $n \to \infty$:

Remark 1 *In the case of $n = 1$ or $n \to \infty$, the function Ψ is:*

$$\Psi\,(1\,,\,\ell) = \frac{1}{\ell^2}\left[arcosh\left(\frac{1}{1-\ell}\right)^2 \frac{1}{2} - \ell\ arcosh\left(\frac{1}{1-\ell}\right)\ coth\left(arcosh\left(\frac{1}{1-\ell}\right)\right)\right], \tag{13}$$

$$\Psi\,(n\,,\,\ell) \to -\frac{log\,(1-\ell)+\ell}{\ell^2}\,, \quad as\ n \to \infty \tag{14}$$

Fixing a value of n, using Remark 1 and the top panel of Figure 1, it can be seen that the menu cost is increasing in ℓ. This is quite intuitive: a larger fraction of free adjustments ℓ requires a higher menu cost, since firms must choose not to adjust prices even when the current price is far away from their ideal price. Indeed equation (11) shows that as $\ell \to 1$, the implied menu cost diverges to $+\infty$. This is also expected, since in the limit case of our version of *Calvo*'s model, the menu cost must prevent any price change, but since the underlying shocks are assumed to follow a random walk in some instances there are arbitrarily large benefits of changing prices. On the other hand, for $\ell = 0$ and $n = 1$, our version of *Golosov-Lucas*'s model, attains its smallest value, which is strictly positive.

Finally, we comment on the effect of n on the implied fixed cost. For a fixed proportion ℓ, the implied per good fixed cost ψ/n is not monotone in the number of products n. Indeed, as stated in equation (11) for a very small share ℓ the values of ψ/n are increasing in n. On the other hand, for larger value of the share ℓ, the order of the implied fixed cost is reversed. Indeed as ℓ goes to 1, equation (12) shows that the ratio of the menu cost per good is lower for higher number of products, and that the ratio between the cost ψ/n between and economy with $n = 1$ and an economy with $n \to \infty$ diverges towards $+\infty$ (this is not obvious because, as shown above, in both economies the implied menu cost ψ/n diverges to ∞.)

Under the view that a plausible model should have a value of ψ/n that is not too high, then the limit of $\ell = 1$, which gives Calvo pricing, should be discarded since the implied cost of a single price adjustment becomes extremely large (it diverges to $+\infty$). Nevertheless, we find it interesting to assess how large are the implied menu costs for models that are close to Calvo pricing, i.e. for ℓ close to one. In this regard, we view the result that for large n the (relative) implied cost is smallest as $\ell \approx 1$ as an argument to favor large values of n. The top panel of Figure 4 shows that –given a $Std(\Delta p_i) = 0.1$, $N_a = 1.5$ and a markup of 25%– the implied menu cost of one price change is a little above 15% of annual profits if $\ell \approx 0.95$ and $n \to \infty$, compared to the 25% level in the case of $n = 1$.

The model also has clear predictions about the per period (say yearly) cost of price adjustments borne by the firms: $(1-\ell)\, N_a\, \psi/n$. In spite of the fact that the cost of a single deliberate price adjustment diverges as $\ell \to 1$, the total yearly cost of adjustment converge to zero continuously. This can be seen in the bottom panel of Figure 4 and is shown analytically in the limiting case of $n \to \infty$: equation (14) gives a simple expression showing that the cost of one price adjustment is monotonically increasing in ℓ, while the total flow cost of adjustment $\Psi(n, \ell)\,(1-\ell)$ is monotonically decreasing.

A simple transformation gives the yearly cost of price adjustments as a fraction of revenues: $\frac{(1-\ell)N_a \;\; \psi/n}{\eta}$, where the scaling by η transforms the units from fraction of profits into fraction of revenues.[33] This statistic is useful because it has empirical counterparts, studied e.g. by Levy et al. (1997). Using equation (10) and the previous definition for the markup yields

$$\frac{\text{Yearly costs of price adjustment}}{\text{Yearly revenues}} = \frac{1}{2}\, \frac{Var(\Delta p_i)}{\mu}\; (1-\ell)\, \Psi\,(n\,,\, \ell) \tag{15}$$

Figure 4 plots the two cost measures in equation (10) and (15) as functions of ℓ, n for an economy with $N_a = 1.5$, $Std(\Delta p_i) = 0.10$ and a markup $\mu \cong 25\%$ (i.e. $B = 10$). We see this parametrization as being consistent with the US data on price adjustments, markups, and the size distribution of price changes discussed above. Alternative parametrizations are readily computed, as discussed below. The figure illustrates how observations on the costs of price adjustments can be used to parametrize the model. Levy et al. (1997) and Dutta et al. (1999) (Table IV and Table 3, respectively) document that for multi-product stores (a handful of supermarkets chain and one drugstore chain) the average cost of price adjustment is around 0.7 percent of revenues. For an economy with $n = 10$ (a reasonable parametrization to fit the size-distribution of price changes) the bottom panel of the figure shows that the model reproduces the yearly cost of 0.7% of revenues when the fraction of free adjustments ℓ is around 60%. The upper panel indicates that at this level of ℓ the cost of one price adjustment is around 5% of profits.

4 The aggregate response to a monetary shock

In this section we give a description of the impulse response of prices and output to an unexpected (once and for all) increase of the money supply of size δ, starting from a steady state with zero inflation. We first describe the general equilibrium set up which is, essentially, the one in Golosov and Lucas (2007), adapted to a multi-product firm as in Alvarez and Lippi (2013b). Then we describe the impulse response of prices and output to a monetary

[33] Since $R = \eta\, \Pi$ where R is revenues per good and Π profits per good.

shock. Our objective is to characterize the impulse response of the aggregate price level as a function of 4 fundamental parameters: the frequency and scale of price changes, N_a and $Std[\Delta p_i]$ respectively, the number of products in the firm's bundle n, and the fraction of free adjustments ℓ.[34]

General Equilibrium Setup. Briefly, this is an economy where each firm produces n goods, each with a linear labor only technology subject to independent idiosyncratic productivity shocks, whose logs follows a BM with instantaneous variance σ^2. As in the previous sections, a firm is subject to a random menu cost to simultaneously change the price of its n products. In a period of length dt this cost equals ψ_ℓ units of labor with probability $1 - \lambda dt$, or zero. Also each firm faces a demand with constant elasticity $\eta > 1$ for each of its n products, coming from households's CES utility function for the consumption aggregate. The $p_i(t)$ in our previous sections are the logs of the markups in each product of the firm relative to the static optimal markup, and our quadratic objective function can be taken to be a second order expansion on the firm's profits with $B = (1/2)\eta(\eta - 1)$. Households' period utility function is additively separable: log in real balances, linear in leisure, and has constant intertemporal elasticity of substitution $1/\epsilon$ for the consumption aggregate.

4.1 The price response to a monetary shock

The initial conditions are the steady state of the economy with constant money supply, and hence constant economy wide price index. The mechanics of the impulse response is that in this economy nominal wages jump on impact by the same percentage as money supply. In Proposition 7 of Alvarez and Lippi (2013b) we show that, up to first order, the firm's optimal policy is to keep $\bar{y}$ unaltered during the transition, a result that can be extended to the present case with $\lambda > 0$. Given this result the characterization of the impulse response is an exercise in aggregation: the steady state distribution of price gaps is perturbed by the common increase in cost across all firms, which will return to steady state slowly, in a process which we describe below. Letting $\mathcal{P}(t)$ be the impulse response of the percentage change in aggregate price level at horizon t, then the percentage change in output is proportional to $\delta - \mathcal{P}(t)$, where the constant of proportionality is $1/\epsilon$.

To compute the IRF of the aggregate price level we find the contribution to the aggregate

[34] Caballero and Engel (2007) perform a related exercise using the Caplin Spulber (S,s) model augmented with a random opportunity of price change, which occurs at rate λ. They study how increasing λ affects the response of the price level (see their Figure 3). There are two channels through which this works: by affecting the frequency of price adjustment and by changing the size and mass of price adjusters. One important difference with respect to them is that our comparative static analysis of a higher λ is done keeping the frequency of adjustment (as well as the scale of the distribution of price changes) constant.

price level of each firm at the time of the shock. They start with price gaps distributed according to g, the invariant distribution. Then the monetary shock displaces them, by subtracting the monetary shock δ to each of them. After that we divide the firms in two groups. Those that adjust immediately and those that adjust at some future time. Note that, for each firm in the cross section, it suffices to keep track only of the contribution to the aggregate price level of the first adjustment after the shock because after that one the future contributions are all equal to zero in expected value. Now we develop the notation to define the impulse response of the aggregate price level.

Let $g\left(p; n, \lambda/\sigma^2, \bar{y}\right)$ be the density of firms with price gap vector $p = (p_1, ..., p_n)$ at time $t = 0$, just before the monetary shock, which corresponds to the invariant distribution with constant money supply. The density g equals the density f of the steady state square norms of the price gaps given by Lemma 2 evaluated at $y = p_1^2 + \cdots + p_n^2$ times a correction for area of sphere and the different variables.[35] In particular we have

$$g\left(p_1, ..., p_n\, ; n, \frac{\lambda}{\sigma^2}, \bar{y}\right) = f\left(p_1^2 + \cdots + p_n^2\, ; n, \frac{\lambda}{\sigma^2}, \bar{y}\right) \frac{\Gamma\left(n/2\right)}{\pi^{n/2}\ \left(p_1^2 + \cdots + p_n^2\right)^{(n-2)/2}}$$

To define the impulse response we introduce two extra pieces of notation. First we let $\{(\bar{p}_1(t,p), \ldots, \bar{p}_n(t,p))\}$ the process for n independent BM, each one with variance per unit of time equal to σ^2, which at time $t = 0$ start at p, so $\bar{p}_i(0,p) = p_i$. We also define the stopping time $\tau(p)$, also indexed by the initial value of the price gaps p as the minimum of two stopping times, τ_1 and $\tau_2(p)$. The stopping time τ_1 denotes the first time since $t = 0$ that jump occurs for a Poisson process with arrival rate λ per unit of time. The stopping time $\tau_2(p)$ denotes the first time that $||\bar{p}(t,p)||^2 > \bar{y}$. Thus $\tau(p)$ is the first time a price change occurs for a firm that starts with price gap p at time zero. The stopped process $\bar{p}(\tau(0), p)$ is the vector of price gaps at the time of price change for such a firm.

The impulse response for the aggregate price level, of which Figure 5 displays several cases, can be written as:

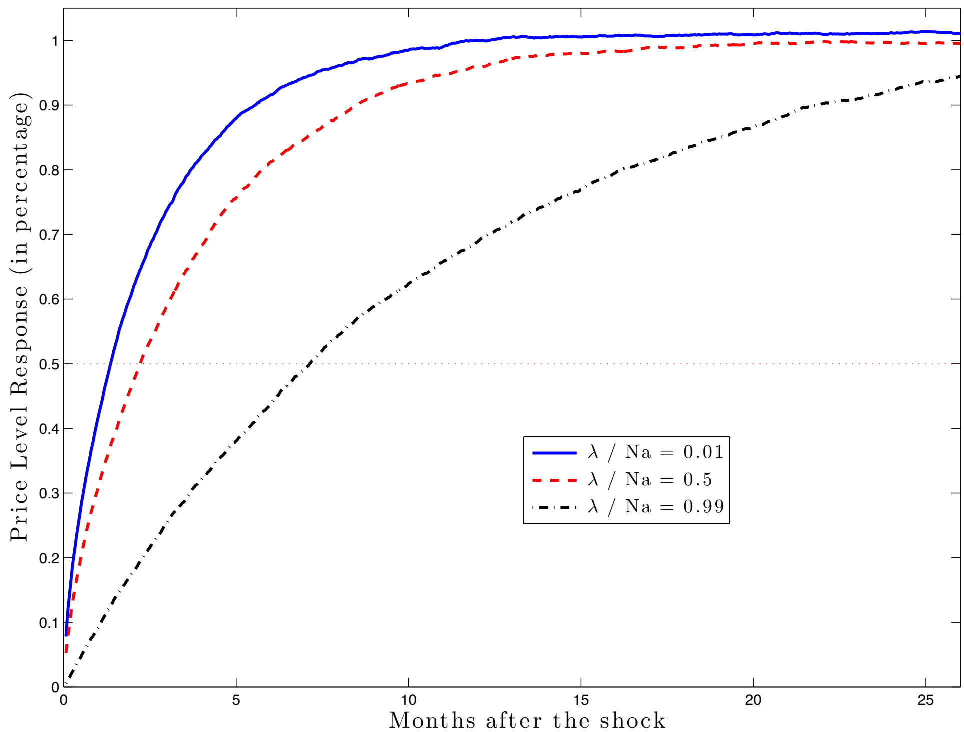
$$\mathcal{P}(t, \delta; \sigma, \lambda, \bar{y}) \;\; = \;\; \Theta(\delta; \sigma, \lambda, \bar{y}) \; + \; \int_0^t \theta(\delta, s; \sigma, \lambda, \bar{y})\, ds\,, \tag{16}$$

where $\Theta(\delta)$ gives the impact effect, the contribution of the monetary shock δ to the aggregate price level on impact, i.e. at the time of the monetary shock. The integral of the θ's gives the remaining effect of the monetary shock in the aggregate price level up to time t, i.e. $\theta(\delta, s)ds$ is the contribution to the increase in the average price level in the interval of times $(s, s + ds)$ from a monetary shock of size δ. Instead the functions θ and Θ are easily defined in terms

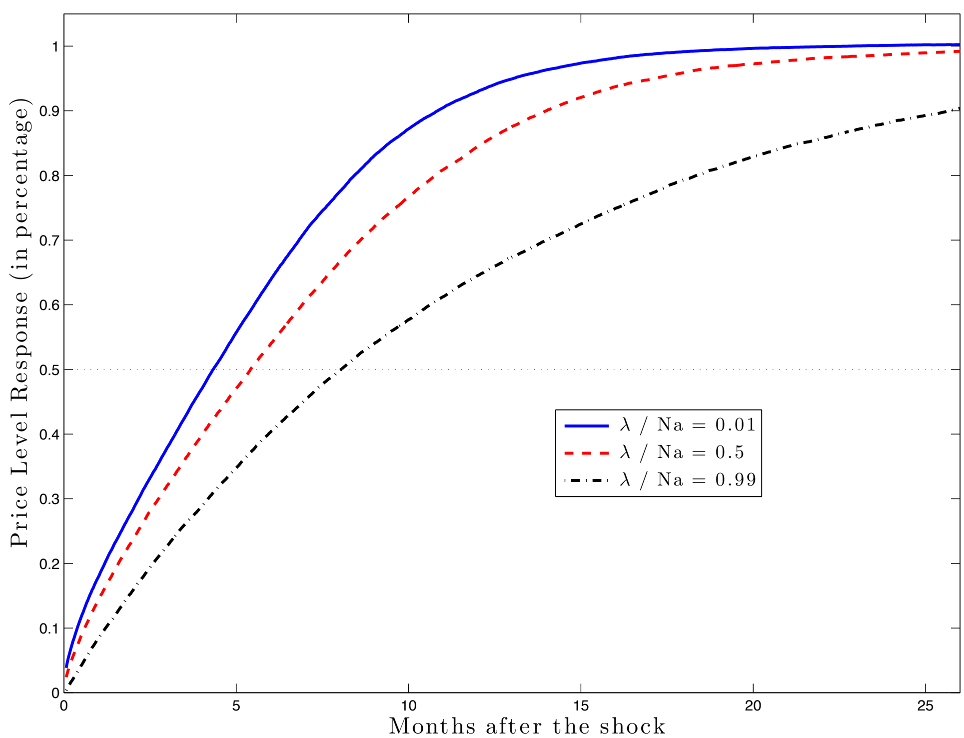
[35]See Section 5 of Alvarez and Lippi (2013b) for this result and the Online appendix for a derivation.

Figure 5: CPI response to a monetary shock of size $\delta = 1\%$

Economy with $n = 1$



Economy with $n = 10$



The figures represent an economy with $N_a = 1.0$ and $std(\Delta p_i) = 0.10$.

of the density g, the process $\{\bar{p}\}$ and the stopping times τ:

$$\Theta(\delta;\sigma,\lambda,\bar{y}) \equiv \int_{||p(0)-\iota\delta||\geq\bar{y}} \left(\delta - \frac{\sum_{j=0}^{n} p_j(0)}{n}\right) g\left(p(0); n, \frac{\lambda}{\sigma^2}, \bar{y}\right) dp_1(0)\cdots dp_n(0)$$

and $\theta(\delta, t;\sigma,\lambda,\bar{y})$ is the density, i.e. the derivative with respect to t of the following expression:

$$\int_{||p(0)-\iota\delta||<\bar{y}} \mathbb{E}\left[-\frac{\sum_{j=0}^{n} \bar{p}_j\left(\tau(p), p\right)}{n} \mathbf{1}_{\{\tau(p)\leq t\}} \,\Big|\, p = p(0) - \iota\delta\right] g\left(p(0); n, \frac{\lambda}{\sigma^2}, \bar{y}\right) dp_1(0)\cdots dp_n(0)$$

where ι is a vector of n ones. This expression takes each firm that has not adjusted price on impact, i.e. those with $p(0)$ satisfying $||p(0)-\iota\delta|| < \bar{y}$, weights them by the relevant density g, displaces the initial price gaps by the monetary shock, i.e. sets $p = p(0) - \iota\delta$, and then looks a the (negative) of the average price gap at the time of the first price adjustment, $\tau(p)$, provided that the price adjustment has happened before or at time t. We make a few remarks about this expression. First, price changes equal the negative of the price gaps because price gaps are defined as prices minus the ideal price. Second, we define θ as a density because, strictly speaking, there is no effect on the price level due to price changes at *exactly* time t, since in continuous time there is a zero mass of firms adjusting at any given time. Third, we can disregard the effect of any subsequent adjustment because each of them has an expected zero contribution to the average price level. Fourth, the impulse response is based on the steady-state decision rules, i.e. adjusting only when $y \geq \bar{y}$ even after an aggregate shock occurs.

Given the results in Proposition 3 -Proposition 4 we can parametrize our model either in terms of $(n, \lambda, \sigma^2, \psi/B)$ or instead parametrize it, for each n, in terms of the implied observable statistics $(N_a, Std[\Delta p_i], \ell)$. These propositions show that this mapping is indeed one-to-one and onto. We refer to ℓ as an "observable" statistic, because we have shown that the "shape" of the distribution of price changes depends only on it.

Proposition 6 *Fix an economy whose firms produce n products and with steady state statistics $(N_a, Std[\Delta p_i], \ell)$. The cumulative proportional response of the aggregate price level $t \geq 0$ periods after a once and for all proportional monetary shock of size δ can be obtained from the one of an economy with one price change per period and with unitary standard deviation of price changes as follows:*

$$\mathcal{P}_{n,\ell}\left(t\,,\,\delta\,;\,N_a\,,\,Std[\Delta p_i]\right) = Std[\Delta p_i]\ \mathcal{P}_{n,\ell}\left(t\,N_a\,,\,\frac{\delta}{Std[\Delta p_i]}\,;\,1\,,\,1\right)\ . \tag{17}$$

This proposition extends the result of Proposition 8 in Alvarez and Lippi (2013b) to the case of $\ell \equiv \lambda/N_a > 0$.[36] The proposition establishes that the shape of the impulse response is completely determined by 2 parameters: n and ℓ, whose comparative static is explored in Figure 5. Economies sharing these parameters but differing in terms of N_a or $Std[\Delta p_i]$ are immediately analyzed by rescaling the values of the horizontal and/or vertical axis. In particular, a higher frequency of price adjustments will imply that the economy "travels faster" along the impulse response function (this is the sense of the rescaling the horizontal axis). Instead, the effect of a larger dispersion of price changes is seen by rescaling the monetary shock δ by $Std[\Delta p_i]$ and by a proportional scaling of the vertical axis. A further simplification to the last result is given by next corollary, showing that for small values of the monetary shocks one can overlook the scaling by $Std[\Delta p_i]$ so that, for a given n and ℓ determining the shape, the most important parameter is the frequency of price changes N_a:

Corollary 1 *For small monetary shocks $\delta > 0$, the impulse response is independent of $Std[\Delta p_i]$. Differentiating equation (17) gives:*

$$\mathcal{P}_{n,\ell}\,(t\,,\,\delta\,;\,N_a\,,\,Std[\Delta p_i]\,) = \delta\,\frac{\partial}{\partial\delta}\,\mathcal{P}_{n,\ell}\,(t\,N_a\,,\,0\,;\,1\,,\,1\,) + o(\delta)$$

for all $t > 0$ and, since $f(\bar{y}) = 0$, then the initial jump in prices can be neglected, i.e.:

$$\mathcal{P}_{n,\ell}\,(0\,,\,\delta\,;\,N_a\,,\,Std[\Delta p_i]\,) \equiv \Theta = o(\delta)\ .$$

4.2 The cumulated output effect of a monetary shock

We provide an analytical characterization of a summary measure for the effect of monetary shocks. The summary measure we choose is the area under the impulse response for output, i.e. the cumulative sum of the output above the steady-state level after a monetary shock of size $\delta > 0$. This measure is closely related to the output variance due to monetary shocks, which is sometimes used in the literature (for more discussion and evidence on this equivalence see the discussion in footnote 21 of Nakamura and Steinsson (2010)). We denote the cumulated output effect as:

$$\mathcal{M}_{n,\ell}(\delta) = (1/\epsilon)\int_0^\infty \left[\delta - \mathcal{P}_{n,\ell}(\delta, t)\right]\,dt \tag{18}$$

[36] The proof in Alvarez and Lippi (2013) is constructive in nature, exploiting results from applied math on the characterization of hitting times for BM in hyper-spheres, which is not longer valid for $\lambda > 0$. Here we use a different strategy which relies on limits of discrete-time, discrete state approximations.

where $1/\epsilon$ is the intertemporal elasticity of consumption, and where $\mathcal{P}_{n,\ell}(\delta, t)$ is the cumulative effect of monetary shock δ on the (log) of the price level after t periods (for notation simplicity we omit the 2 scaling parameters from the $\mathcal{P}_{n,\ell}$ function). We consider here the case of small shocks δ, a realistic standard in this literature, by taking the first order approximation to equation (18), so we consider $\mathcal{M}_{n,\ell}(\delta) \approx \mathcal{M}'_{n,\ell}(0)\delta$.[37] As a consequence of our characterization in Proposition 6, the derivative $\mathcal{M}'_{n,\ell}(0)$ is the product of $1/(\epsilon\, N_a)$ times a function that depends on n and ℓ only. The scaling for N_a is quite intuitive: the effect of monetary policy is inversely proportional to price flexibility. Likewise the scaling by $1/\epsilon$ reflects the intertemporal elasticity of consumption. The next result links the scaled effect of monetary policy with the kurtosis of the price changes.

Proposition 7 *Consider an economy whose firms produce $n > 1$ products, with steady-state statistics $(N_a, Std[\Delta p_i], \ell)$ and a steady-state kurtosis of price changes $kurt_{n,\ell}(\Delta p_i)$. Then:*

$$\mathcal{M}_{n,\ell}(\delta; N_a, Std[\Delta p_i]) = \delta\ \mathcal{M}'_{n,\ell}(0; N_a, 1) + o(\delta) = \frac{\delta}{\epsilon\, N_a}\ \frac{kurt_{n,\ell}(\Delta p_i)}{6}\ + o(\delta)$$

This proposition is quite useful to explain what produces the different results that are found in the literature on the real effects of monetary policy. The proposition illustrates how it is possible for two models sharing similar features, e.g. calibrated to the same observables $N_a, Std(\Delta p_i)$ and sharing the same preference specification (substitution elasticity), to produce different predictions about the output effect: what is needed is that the model predicts a different kurtosis of price changes.

Recall from Proposition 4 that the shape of the size distribution of price changes, and hence kurtosis, depends only on n and ℓ. For a fixed n, kurtosis is increasing in ℓ. Indeed, as ℓ goes to 1 then kurtosis goes to 6, and hence we obtain $\mathcal{M}_{n,\ell}(\delta) \cong \delta/(\epsilon N_a)$, which is the result produced by the Calvo pricing model. On the other extreme, as $\ell = 0$ we have that kurtosis equals $3n/(n+2)$. This implies that, for instance, in the Golosov and Lucas case of $n = 1$, the impact of monetary policy is 1/6 of Calvo. Also, keeping $\ell = 0$ and varying n from 1 to ∞, the effect goes from 1/6 to 1/2 of Calvo. Note that in the case of $\ell = 0$ and $n = \infty$ the model becomes Taylor's staggered price model or, equivalently, Reis (2006) model. The reason why the purely multiproduct-Taylor-Reis case ($\ell = 0$ and $n = \infty$) delivers only half of the effect of a monetary shock than the purely random menu cost-Calvo case ($\ell = 1$) is that the latter has a more persistent effect. Interestingly, neither model features the selection effect typical of the Golosov and Lucas model, i.e. in both cases the average price change at any time after a monetary shock equals the size of the shock δ. Yet, the models differ in their persistence, the timing of the adjustments in the case of $\ell = 0$ and $n = \infty$ is faster, as

[37] As common to menu cost problems the model displays a substantial price flexibility for large shocks.

seen in the linear shape of the impulse response, while the $\ell = 1$ case features an exponential impulse response. This shape is even evident in Figure 5 for the panel with the $n = 10$ case, comparing across values of ℓ.

The discussion above makes clear that the assumption of non-gaussian shocks in Midrigan (2011) is quite crucial to obtain real effects that are closer to Calvo than to Golosov-Lucas. What is needed for the effects to be large is a large kurtosis, which Midrigan obtains by assuming a process for the shocks hitting the firm's costs that are fat-tailed. It can indeed be shown that introducing fat-tails in our version of the Golosov-Lucas model, through shocks to the marginal cost occurring with a Poisson intensity, leads to a formally similar problem to that of the model with free adjustment opportunities.

Figure 6: Cumulated output effect relative to Calvo pricing: $kurt_{n,\ell}(\Delta p_i)$

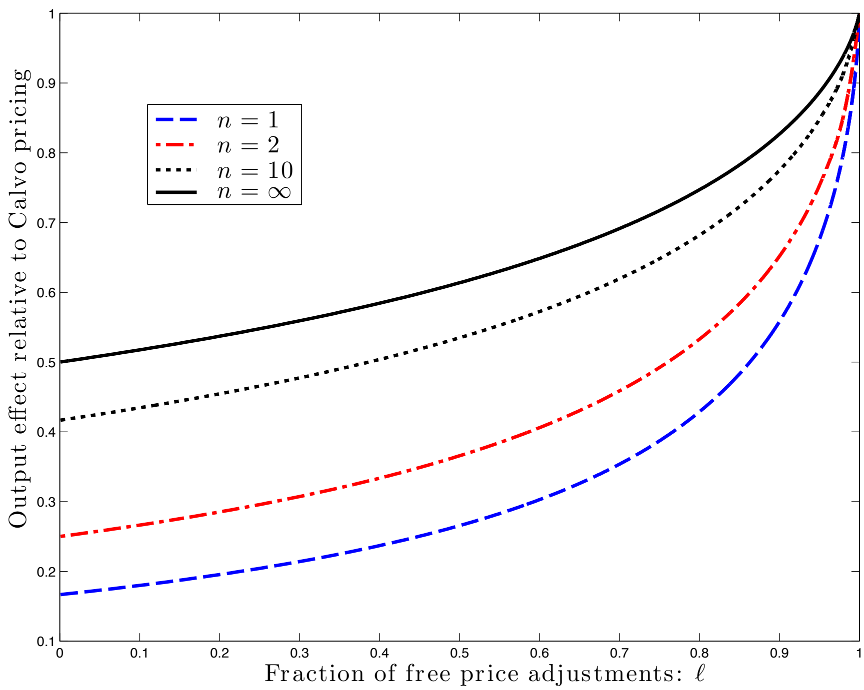


Figure 6 offers a richer systematic comparison of the real effects of monetary shocks as n and ℓ vary: the vertical axis plots the real output effect produced by a small monetary shock relative to the effect produced by a Calvo model where $\ell = 1$. Four curves are plotted in the figure, corresponding to $n = 1, 2, 10, \infty$. It appears that the model behavior for $n = 2$ remains quite close to the case where $n = 1$, as was also seen from the analysis of the distribution of price changes. Instead, the model behavior for $n = 10$ is quite close to that of a model where $n = \infty$. This is useful because the latter is quite tractable analytically, as discussed below. Figure 6 shows that at any level of ℓ the real output effect are smallest for $n = 1$. As explained in Alvarez and Lippi (2013b) a larger number of goods dampens the selection effect of monetary policy increasing the real output consequences of a monetary shock. Indeed at any level of ℓ the effect is increasing in n. The figure shows that fixing n the output effect is increasing in ℓ. In the limit, as $\ell \to 1$ the economy converges to a Calvo model where the real effects are largest and independent of n.

It is interesting to notice that the curves plotted in the figure are convex. In particular, some analysis (see the formula for $\mathcal{M}$ in the special case when $n = \infty$ below) reveals that the slope of the curve as $\ell \to 1$ diverges to $+\infty$ for any level of n. The economic implication of this property is that a small deviation from Calvo pricing, i.e. a fraction of adjustment ℓ that is slightly below 1 is going to give rise to a large deviation from the real effects predicted by the Calvo pricing. That the relatively large real effects in Calvo are very sensitive to the introduction of a small amount of selection by firms regarding the timing of price changes, is also apparent in the CalvoPlus model of Nakamura and Steinsson (2010) (see their figure VII). Hence the finding seems robust as these models, and measure of real effects, are similar but not identical.

Aggregation. Assume that there are S sectors, each with an expenditure weight $e(s) > 0$, and with different parameters so that each have $N_a(s)$ price changes per unit of time, and a distribution of price changes with kurtosis $kurt(s)$. In this case, after repeating the arguments above for each sector and aggregating, we obtain that the area under the IRF of *aggregate* output for a small monetary shock δ is

$$\mathcal{M}(\delta) = \frac{1}{6}\frac{\delta}{\epsilon} \sum_{s\in S} \frac{e(s)}{N_a(s)} kurt(s) = \frac{1}{6}\frac{\delta}{\epsilon}\, D \sum_{s\in S} d(s)\, kurt(s) \tag{19}$$

where D is the expenditure-weighted average duration of prices $D \equiv \sum_{s\in S} \frac{e(s)}{N_a(s)}$ and the $d(s) \equiv \frac{e(s)}{N_a(s)D}$ are weights taking into account both relative expenditures and durations. In the case in which all sectors have the same durations then $d(s) = e(s)$ and $\mathcal{M}$ is proportional to the kurtosis of the standardized data. Likewise, the same result applies if all sectors have the same kurtosis.[38] In general, if sectors are heterogenous in the durations (or expenditures), then the kurtosis of the sectors with longer duration (or expenditures) receive a higher weight in the computation of $\mathcal{M}$. For the French data, computation of the duration weighted kurtosis in equation (19) results in an increase of the order 15%, reflecting a correlation between kurtosis and duration of the same magnitude.

[38] The effect of heterogeneity in N_a on aggregation is well known, so that D is different from the average of N_a's, see for example Carvalho (2006) and Nakamura and Steinsson (2010).

The output effect in two limiting cases: $n = 1$ and $n = \infty$. We conclude by discussing two limiting cases for which a tractable closed form expression can be derived which bracket the possible range of output effects. For each case we derive the implications for the cumulated output effect while considering the full range of values for $\ell \in (0, 1)$ and keeping the frequency and variance of price changes constant.

The expression for these effects are given by (see Appendix H for a derivation)

$$\mathcal{M}'(0) = \begin{cases} \frac{1}{\epsilon N_a} \frac{\left(e^{\sqrt{2\phi}}+e^{-\sqrt{2\phi}}\right)\left(e^{\sqrt{2\phi}}+e^{-\sqrt{2\phi}}-2(1+\phi)\right)}{\left(e^{\sqrt{2\phi}}+e^{-\sqrt{2\phi}}-2\right)^2} & \text{where} \quad \ell = \frac{e^{\sqrt{2\phi}}+e^{-\sqrt{2\phi}}-2}{e^{\sqrt{2\phi}}+e^{-\sqrt{2\phi}}} \quad \text{and} \quad n = 1 \\ \\ \frac{1}{\epsilon N_a} \left(\frac{1-(1+\phi)e^{-\phi}}{\left(1-e^{-\phi}\right)^2} \right) & \text{where} \quad \ell = 1 - e^{-\phi} \quad \text{and} \quad n \to \infty \end{cases}$$

where different values of ϕ map monotonically into the fraction of free adjustments ℓ as shown in the display. We find the $n \to \infty$ case interesting because of its tractability and because, as shown in Figure 6, it provides a benchmark for the cases where n is high (e.g. $n \cong 10$).

5 Concluding remarks

The paper offered new evidence on price setting behavior using different microeconomic data. The patterns reveal that after correcting for time invariant cross section heterogeneity and measurement error the size-distribution of price changes in France has a shape in between the Normal and the Laplace distribution. In words, it displays more small as well as more large price changes than a Normal distribution. Similar patterns, although perhaps closer to the Normal, appear in the US data.

The paper developed a theoretical model that is able to qualitatively reproduce these cross sectional patterns. The model can be calibrated to the data by matching 4 moments, all of which can be disciplined by empirical observations on the frequency, scale and kurtosis of price changes, as well as a measure of the costs of price adjustment. The model provides an analytical characterization of the propagation of the monetary shocks and illustrates how the propagation depends on the fundamental parameters. We see this as progress because previous contributions rely on numerical solutions which are less apt in identifying the key causes of the effects, as in e.g. Caballero and Engel (2007); Golosov and Lucas (2007); Dotsey, King, and Wolman (2009); Midrigan (2011).

Our analytical model nests several classic models of price setting, as Taylor (1980); Calvo (1983); Reis (2006); Golosov and Lucas (2007); Midrigan (2011); Alvarez and Lippi (2013b), and allows for an immediate comparison showing what empirical observations are useful to select among them. In particular, a somewhat surprising result is that the real cumulated output effect of a monetary shock is proportional to the kurtosis of the size-distribution of price changes. The model thus suggests that a precise measurement of kurtosis is key to answer the question on the size of the real effects of monetary policy, and explains the large differences predicted by the previous model largely in terms of their different predictions for kurtosis. For instance if the kurtosis of price changes is 6, as under a Laplace distribution of price changes, the real effects of monetary policy are the ones produced by a Calvo model. If the kurtosis is 3, as under a Normal distribution, the real effects of monetary policy are the ones produced by a Taylor model, which are half of the ones in Calvo. Our empirical evidence for France suggested that a kurtosis that is between 4 and 5, which implies real effects of monetary policy that are 4 times larger than in the classic menu cost model of Golosov and Lucas (2007), but 20 to 30% smaller than predicted by the widely used Calvo's price setting mechanism. For the US, where the size distribution of price changes appears closer to a Normal (once measurement error is accounted for), the effect seems to be about 50% smaller than in the Calvo model, very close to the effect predicted by a Taylor model. Our results suggest an area for additional empirical effort is to precisely measure kurtosis on various datasets, as this moment appears crucially related to the real effects of monetary policy in a variety of models, and yet its measurement is notoriously sensitive to outliers and sampling errors.

References

Alvarez, Fernando E. and Francesco Lippi. 2013a. "The demand of liquid assets with uncertain lumpy expenditures." *Journal of Monetary Economics* 60 (7):753–770.

———. 2013b. "Price setting with menu costs for multi product firms." *Econometrica* forthcoming.

Alvarez, Luis J., Emmanuel Dhyne, Marco Hoeberichts, Claudia Kwapil, Herve Le Bihan, Patrick Lunnemann, Fernando Martins, Roberto Sabbatini, Harald Stahl, Philip Vermeulen, and Jouko Vilmunen. 2006. "Sticky Prices in the Euro Area: A Summary of New Micro-Evidence." *Journal of the European Economic Association* 4 (2-3):575–584.

Barro, Robert J. 1972. "A Theory of Monopolistic Price Adjustment." *Review of Economic Studies* 39 (1):17–26.

Baudry, L., H. Le Bihan, P. Sevestre, and S. Tarrieu. 2007. "What do Thirteen Million Price Records have to Say about Consumer Price Rigidity?" *Oxford Bulletin of Economics and Statistics* 69 (2):139–183.

Berardi, Nicoletta, Erwan Gautier, and Hervé Le Bihan. 2013. "More facts about prices: France before and during the great recession." Working Paper 425, Banque de France.

Burstein, Ariel and Christian Hellwig. 2006. "Prices and market shares in state-dependent pricing models." 2006 Meeting Papers 375, Society for Economic Dynamics.

Caballero, Ricardo J. and Eduardo M. R. A. Engel. 1999. "Explaining Investment Dynamics in U.S. Manufacturing: A Generalized (S,s) Approach." *Econometrica* 67 (4):783–826.

Caballero, Ricardo J. and Eduardo M.R.A. Engel. 2007. "Price stickiness in Ss models: New interpretations of old results." *Journal of Monetary Economics* 54 (Supplement):100–121.

Calvo, Guillermo A. 1983. "Staggered prices in a utility-maximizing framework." *Journal of Monetary Economics* 12 (3):383–398.

Carvalho, Carlos. 2006. "Heterogeneity in Price Stickiness and the Real Effects of Monetary Shocks." *The B.E. Journal of Macroeconomics* 6 (3):1–58.

Cavallo, Alberto. 2010. "Scraped Data and Sticky Prices." Tech. rep., MIT Sloan.

Chakrabarti, Rajesh and Barry Scholnick. 2007. "The mechanics of price adjustment: new evidence on the (un) importance of menu costs." *Managerial and Decision Economics* 28 (7):657–668.

Chen, Haipeng, Daniel Levy, Sourav Ray, and Mark Bergen. 2008. "Asymmetric price adjustment in the small." *Journal of Monetary Economics* 55 (4):728–737.

Christopoulou, Rebekka and Philip Vermeulen. 2012. "Markups in the Euro area and the US over the period 19812004: a comparison of 50 sectors." *Empirical Economics* 42 (1):53–77.

Costain, James and Anton Nakov. 2011. "Price Adjustments in a General Model of State–Dependent Pricing." *Journal of Money, Credit and Banking* 43:385–406.

———. 2012. "Distributional dynamics under smoothly state-dependent pricing." *Journal of Monetary Economics* forthcoming.

Dixit, Avinash. 1991. "Analytical Approximations in Models of Hysteresis." *Review of Economic Studies* 58 (1):141–51.

Dotsey, Michael and Robert G. King. 2005. "Implications of state-dependent pricing for dynamic macroeconomic models." *Journal of Monetary Economics* 52 (1):213–242.

Dotsey, Michael, Robert G. King, and Alexander L. Wolman. 1999. "State-Dependent Pricing And The General Equilibrium Dynamics Of Money And Output." *The Quarterly Journal of Economics* 114 (2):655–690.

———. 2009. "Inflation and Real Activity with firm-level productivity shocks." Tech. rep., Federal Reserve Bank of Richmond.

Dutta, Shantanu, Mark Bergen, Daniel Levy, and Robert Venable. 1999. "Menu Costs, posted prices and multiproduct retailers." *The Journal of Money, Credit and Banking* 31 (4):683–703.

Eichenbaum, Martin, Nir Jaimovich, and Sergio Rebelo. 2011. "Reference Prices, Costs, and Nominal Rigidities." *American Economic Review* 101 (1):234–62.

Eichenbaum, Martin S., Nir Jaimovich, Sergio Rebelo, and Josephine Smith. 2013. "How Frequent Are Small Price Changes?" *American Economic Journal: Macroeconomics* forthcoming.

Golosov, Mikhail and Robert E. Jr. Lucas. 2007. "Menu Costs and Phillips Curves." *Journal of Political Economy* 115:171–199.

Kehoe, Patrick J. and Virgiliu Midrigan. 2007. "Sales and the real effects of monetary policy." Working Papers 652, Federal Reserve Bank of Minneapolis.

Kiley, Michael. 2002. "Partial Adjustment and Staggered Price Setting." *Journal of Money, Credit and Banking* 34 (2):283–98.

Klenow, J. Peter and Benjamin Malin. 2010. "Microeconomic Evidence on Price-Setting." *Handbook of monetary economics* 3:231–284.

Klenow, Peter J. and Oleksiy Kryvtsov. 2008. "State-Dependent or Time-Dependent Pricing: Does It Matter for Recent U.S. Inflation?" *The Quarterly Journal of Economics* 123 (3):863–904.

Labbé, Felipe. 2013. "Small and large price changes: Microeconomic evidence from Chilean supermakets." manuscript, University of Chicago.

Levy, Daniel, Mark Bergen, Shantanu Dutta, and Robert Venable. 1997. "The Magnitude of Menu Costs: Direct Evidence from Large U.S. Supermarket Chains." *The Quarterly Journal of Economics* 112 (3):791–825.

Midrigan, Virgiliu. 2009. "Menu Costs, Multi-Product Firms, and Aggregate Fluctuations." Working paper, NYU.

———. 2011. "Menu Costs, Multi-Product Firms, and Aggregate Fluctuations." *Econometrica,* 79 (4):1139–1180.

Nakamura, Emi and Jon Steinsson. 2008. "Five Facts about Prices: A Reevaluation of Menu Cost Models." *The Quarterly Journal of Economics* 123 (4):1415–1464.

———. 2010. "Monetary Non-Neutrality in a Multisector Menu Cost Model." *The Quarterly Journal of Economics* 125 (3):961–1013.

Reis, Ricardo. 2006. "Inattentive producers." *Review of Economic Studies* 73 (3):793–821.

Stokey, Nancy L. 2008. *Economics of Inaction: Stochastic Control Models with Fixed Costs.* Princeton University Press.

Taylor, John B. 1980. "Aggregate Dynamics and Staggered Contracts." *Journal of Political Economy* 88 (1):1–23.

Vavra, Joseph. 2013. "Inflation Dynamics and Time-Varying Volatility: New Evidence and an Ss Interpretation." *The Quarterly Journal of Economics* (forthcoming).

Willis, Jonathan L. 2000. "Estimation of adjustment costs in a model of state-dependent pricing." Tech. Rep. RWP 00-07, Federal Reserve Bank of Kansas City.

Woodford, Michael. 2009. "Information-Constrained State-Dependent Pricing." *Journal of Monetary Economics* 56:s100–s124.

Wulfsberg, F. 2010. "Price adjustments and inflation-evidence from Norwegian consumer price data 1975-2004." Working paper, Bank of Norway.

Zaitsev, V.F. and A.D. Polyanin. 2003. *Handbook of Exact Solutions for Ordinary Differential Equations*. Chapman and Hall, CRC Press.

Zbaracki, Mark J., Mark Ritson, Daniel Levy, Shantanu Dutta, and Mark Bergen. 2004. "Managerial and Customer Costs of Price Adjustment: Direct Evidence from Industrial Markets." *The Review of Economics and Statistics* 86 (2):514–533.

A Proofs

Proof. (of Proposition 1)
Let $p(0) = 0$. Define $x(t) \equiv ||p(t)||^2 - n\,\sigma^2\,t$ for $t \geq 0$. Using Ito's lemma we can verify that the drift of $||p||^2$ is $n\sigma^2$, and hence $x(t)$ is a Martingale. By the optional sampling theorem $x(\tau)$, the process stopped at τ, is also a martingale. Then

$$\mathbb{E}\left[x(\tau)\;\middle|\;p(0)\right] = \mathbb{E}\left[||p(\tau)||^2\;\middle|\;p(0)\right] - n\sigma^2\mathbb{E}\left[\tau\;\middle|\;p(0)\right] = x(0) = 0$$

and since

$$N_a = 1/\mathbb{E}\left[\tau\;\middle|\;p(0)\right] \text{ and } Var(\Delta p_i) = \mathbb{E}\left[||p(\tau)||^2\;\middle|\;p(0)\right]/n$$

we obtain the desired result. □

Proof. (of Lemma 1) First, note that since two value functions differ by a constant, then all their derivatives are identical. Hence, if the one for the discount rate and arrival rate of free adjustment $(r+\lambda, 0)$ satisfies value matching and smooth pasting, so does the one for discount rate and arrival rate of free adjustment $(r, \lambda, 0)$, for the same boundary. Second, consider the range of inaction, subtracting the value function for the problem with parameters $(r+\lambda, 0)$ from the one with parameters (r, λ), and using that all the derivatives are identical, one verifies that if the Bellman equation holds for the problem with $(r+\lambda, 0)$, so it does for the problem with (r, λ). □

Proof. (of Proposition 2) The first part is straightforward given Lemma 1 and Proposition 3 in Alvarez and Lippi (2013b). The second part is derived from the following implicit expression determining $\bar{y}$ (see the proof of Proposition 3 in Alvarez Lippi (2013) for a derivation):

$$\psi \quad = \quad \frac{B}{r+\lambda}\bar{y}\left[1-\frac{\frac{2\sigma^2(n+2)}{r+\lambda}\bar{y}+\bar{y}^2+\bar{y}^2\sum_{i=1}^{\infty}\kappa_i\ (r+\lambda)^i\ \bar{y}^i}{\frac{2\sigma^2(n+2)}{r+\lambda}\bar{y}+2\bar{y}^2+\bar{y}^2\sum_{i=1}^{\infty}\kappa_i\ (i+2)\ (r+\lambda)^i\ \bar{y}^i}\right] \tag{20}$$

where $\kappa_i = (r+\lambda)^{-i}\prod_{s=1}^{i}\frac{1}{\sigma^2(s+2)(n+2s+2)}$. So we can write this expression as:

$$\psi = \frac{B}{r+\lambda}\bar{y}\left[1-\xi(\sigma^2, r+\lambda, n, \bar{y})\right]$$

where $\xi(\sigma^2, r+\lambda, n, \bar{y})$ is given by:

$$\xi(\sigma^2, r+\lambda, n, \bar{y}) \equiv \frac{\frac{2\sigma^2(n+2)}{r+\lambda}\bar{y}+\bar{y}^2+\bar{y}^2\sum_{i=1}^{\infty}\kappa_i\ (r+\lambda)^i\ \bar{y}^i}{\frac{2\sigma^2(n+2)}{r+\lambda}\bar{y}+2\bar{y}^2+\bar{y}^2\sum_{i=1}^{\infty}\kappa_i\ (i+2)\ (r+\lambda)^i\ \bar{y}^i}$$

Since $\bar{y}\to\infty$ as $\psi\to\infty$ then we can define the limit:

$$\lim_{\psi\to\infty}\frac{\psi}{\bar{y}} = \frac{B}{r+\lambda}\left[1-\lim_{\bar{y}\to\infty}\xi(\sigma^2, r+\lambda, n, \bar{y})\right]$$

Simple analysis can be used to show that $\lim_{\bar{y}\to\infty}\xi(\sigma^2, r+\lambda, n, \bar{y}) = 0$ which gives the expression in the proposition (see the Online Appendix G for a detailed derivation). $\square$

Proof. (of Proposition 3). To characterize N_a we write the Kolmogorov backward equation for the expected time between adjustments $\mathcal{T}(y)$ which solves: $\lambda\mathcal{T}(y) = 1 + \ n\ \sigma^2\ \mathcal{T}'(y) + 2\ y\ \sigma^2\ \mathcal{T}''(y)$ for $y\in(0,\bar{y})$ and $\mathcal{T}(\bar{y}) = 0$ (see Appendix E for a discussion of the solution to this equation). Then the expected number of adjustments is given by $N_a = 1/\mathcal{T}(0)$, subject to $\mathcal{T}(0) < \infty$.

We guess that the solution of the ODE equation (3) has a power series representation:

$$\mathcal{T}(y) = \sum_{i=0}^{\infty}\ \alpha_i\ y^i\ , \qquad \text{for } y\in[0,\bar{y}]\ . \tag{21}$$

and then obtain the following conditions on its coefficients $\{\alpha_i\}$:

$$\alpha_1 = \frac{\lambda\alpha_0 - 1}{n\sigma^2}\quad ,\quad \alpha_{i+1} = \frac{\lambda}{(i+1)\ \sigma^2\ (n+2i)}\ \alpha_i\ ,\quad \text{for } i\geq 1\ . \tag{22}$$

and where $0 < \alpha_0 < 1/\lambda$ is chosen to that $0 \geq \alpha_i$ for $i\geq 1$, $\lim_{i\to\infty}\frac{\alpha_{i+1}}{\alpha_i} = 0$ and $0 =$

$\sum_{i=0}^{\infty} \alpha_i \; \bar{y}^i$. Moreover, $\mathcal{T}(0) = \alpha_0$ is an increasing function of $\bar{y}$ since α_0 solves:

$$0 = \alpha_0 + \frac{(\alpha_0 - 1/\lambda)}{n} \left(\frac{\bar{y}\lambda}{\sigma^2}\right) \left[1 + \sum_{i=1}^{\infty} \left(\prod_{k=1}^{i} \frac{1}{(k+1)\;(n+2k)}\right) \left(\frac{\bar{y}\lambda}{\sigma^2}\right)^i\right] \quad \text{or}$$

$$\alpha_0 \left\{1 + \frac{1}{n} \left(\frac{\bar{y}\lambda}{\sigma^2}\right) \left[1 + \sum_{i=1}^{\infty} \left(\prod_{k=1}^{i} \frac{1}{(k+1)\;(n+2k)}\right) \left(\frac{\bar{y}\lambda}{\sigma^2}\right)^i\right]\right\}$$

$$= \frac{1}{\lambda n} \left(\frac{\bar{y}\lambda}{\sigma^2}\right) \left[1 + \sum_{i=1}^{\infty} \left(\prod_{k=1}^{i} \frac{1}{(k+1)\;(n+2k)}\right) \left(\frac{\bar{y}\lambda}{\sigma^2}\right)^i\right].$$

Solving for α_0 gives the desired expression. The second order approximation follows from differentiating this expression twice. $\square$

We first state a lemma about the density $f(y)$.

Lemma 2 *Let $f(y; n, \frac{\lambda}{\sigma^2}, \bar{y})$ be the density of $y \in [0, \bar{y}]$ in equation (5) satisfying the boundary conditions. For any $k > 0$*

$$f\left(y; n, \frac{\lambda}{\sigma^2}, \bar{y}\right) = \frac{1}{k}\; f\left(\frac{y}{k}; n, \frac{\lambda k}{\sigma^2}, \frac{\bar{y}}{k}\right)$$

Proof. (of Lemma 2). Consider the function $f(y; n, \frac{\lambda}{\sigma^2}, \bar{y})$ solving equation (5) (and boundary conditions) for given $n, \frac{\lambda}{\sigma^2}, \bar{y}$. Without loss of generality set $\sigma' = \sigma$ and consider $\bar{y}' = \bar{y}/k$ and $\lambda' = \lambda k$. Notice that by setting $C_1' = C_1 k$ and $C_2' = C_2 k$ we verify that the boundary conditions hold (because $C_1'/C_2' = C_1/C_2$) and that (5) holds (which is readily verified by a change of variable). $\square$

Proof. (of Proposition 4) Let $w\left(\Delta p_i; n, \ell, Std(\Delta p_i)\right)$ be the density function in equation (41). Next we verify equation (8). From the first term in equation (41) notice that

$$(1-\ell)\, \omega\left(\Delta p_i; \bar{y}\right) = s\; (1-\ell)\, \omega\left(s\Delta p_i; s^2 \bar{y}\right)$$

where the first equality uses the homogeneity of degree -1 of $\omega(\Delta p_i; y)$ (see equation (39)). From the second term in equation (41) for $n \geq 2$

$$\ell \int_0^{\bar{y}} \omega(\Delta p_i; y) f(y; n, \frac{\lambda}{\sigma^2}, \bar{y}) dy = \ell \int_0^{\bar{y}} s\; \omega\left(s\Delta p_i; s^2 y\right)\;\; s^2\; f\left(ys^2; n, \frac{\lambda}{s^2\sigma^2}, \bar{y}s^2\right) dy$$

where the first equality follows from Lemma 2 for $k = 1/s^2$, and the homogeneity of degree -1 of $\omega(\cdot, \cdot)$. Further we note

$$\ell \int_0^{\bar{y}} s\; \omega\left(s\Delta p_i; s^2 y\right) s^2\, f\left(ys^2; n, \frac{\lambda}{s^2\sigma^2}, \bar{y}s^2\right) dy = s^3 \ell \int_0^{\bar{y}} \omega\left(s\Delta p_i; s^2 y\right) f\left(ys^2; n, \frac{\lambda'}{\sigma'^2}, \bar{y}'\right) dy$$

where we note that $\frac{\lambda' \bar{y}'}{\sigma'^2} = \frac{\lambda \bar{y}}{\sigma^2}$, so that ℓ is the same across the two economies. Using the change of variable $z = y\ s^2$

$$s^3 \ell \int_0^{\bar{y}} \omega\left(s\Delta p_i; s^2 y\right) f\left(y s^2; n, \frac{\lambda'}{\sigma'^2}, \bar{y}'\right) dy = s\ \ell \int_0^{\bar{y}'} \omega\left(s\Delta p_i; z\right) f\left(z; n, \frac{\lambda'}{\sigma'^2}, \bar{y}'\right) dz\ .$$

where $\bar{y}' = s^2 \bar{y}$, which completes the verification of equation (8). $\square$

Proof. (of Proposition 5). To obtain the expression in equation (10) we use the characterization of $\ell = \mathcal{L}_n\left(\frac{\lambda \bar{y}}{n\sigma^2}\right)$ of Proposition 3, it is equivalent to fix a value of $\phi \equiv \frac{\lambda \bar{y}}{n\sigma^2}$. We let the optimal decision rule be $\bar{y}\left(\psi/B, \sigma^2, r+\lambda, n\right)$ so that we have:

$$\bar{y}\left(\frac{\psi}{B}, \sigma^2, r+\lambda, n\right)\ \frac{\lambda}{n\,\sigma^2} = \phi$$

Moreover we have that to be consistent with $Var(\Delta p_i)$ and N_a we have, using Proposition 1 and $\ell = \mathcal{L}(\phi, n)$:

$$N_a = \lambda/\mathcal{L}(\phi, n) \ \text{ and } \frac{\lambda}{\sigma^2} = \mathcal{L}(\phi, n)/Var(\Delta p_i)\ .$$

Thus, after taking $r \downarrow 0$ and using the expression above we can write:

$$\bar{y}\left(\frac{\psi}{B}\,,\ N_a Var(\Delta p_i)\,,\ \mathcal{L}(\phi, n) N_a\,,\ n\right)\ \frac{\mathcal{L}(\phi, n)}{n\,Var(\Delta p_i)} = \phi$$

Fixing n and totally differentiating this expression with respect to $(\psi/B, N_a, Var(\Delta p_i))$, and denoting by η_ψ, η_{σ^2}, η_λ the elasticities of $\bar{y}$ with respect to ψ/B, σ^2, λ we have:

$$\eta_\psi\, \hat{\psi} + \eta_{\sigma^2}\, (\hat{N}_a + \hat{V}ar(\Delta p_i)) + \eta_\lambda\, \hat{N}_a = \hat{V}ar(\Delta p_i)$$

where a *hat* denotes a proportional change. Using Proposition 3-(iv) in Alvarez and Lippi (2013b) and Lemma 1 we have that these elasticities are related by:

$$\eta_\lambda = 2\eta_\psi - 1 \ \text{ and } \ \eta_{\sigma^2} = 1 - \eta_\psi\ .$$

Thus

$$\eta_\psi\, \hat{\psi} + (1-\eta_\psi)\, (\hat{N}_a + \hat{V}ar(\Delta p_i)) + (2\eta_\psi - 1)\, \hat{N}_a = \hat{V}ar(\Delta p_i)\ .$$

Rearranging and canceling terms:

$$\eta_\psi\, \hat{\psi} + \eta_\psi\, \hat{N}_a - \eta_\psi \hat{V}ar(\Delta p_i) = 0\ .$$

Dividing by η_ψ we obtain that $\hat{\psi} = \hat{V}ar(\Delta p_i) - \hat{N}_a$. Additionally, since $\bar{y}$ is a function of B, then we can write $\psi/n = B\,(Var(\Delta p_i)/N_a)\ \Psi(n, \phi)$.

That $\psi \to \infty$ as $\ell \to 1$ follows because $\mathcal{L}(\phi, n) \to 1$ as $\phi \to \infty$ and because, by Proposition 3-(i) in Alvarez and Lippi (2013b), $\bar{y}$ is increasing in ψ and has range and domain $[0, \infty)$. For $\lambda = 0$ and $N_a > 0$ we obtain: $\frac{\psi}{n} = B\ \frac{Var(\Delta p)}{N_a}\ \frac{n}{2\,(n+2)}$ .This follows from using the square root approximation of $\bar{y}$ for small $\psi\,(\lambda + r)^2$, the expression for $N_a = n\sigma^2/\bar{y}$ and Proposition 1, i.e. $N_a\, Var(\Delta p_i) = \sigma^2$.

To obtain the expression for $\Psi(n, 0)$ we use Proposition 6 in Alvarez and Lippi (2013b) where it is shown that for $\lambda = 0$ then $Kur(\Delta p_i) = 3n/(n+2)$.

Finally, for $n = \infty$, we can consider the value function per product, obtaining

$$v = \min_T B \int_0^T \sigma^2 \, t \, e^{-(\lambda+r)t} \, dt + \int_0^T \lambda \, v \, e^{-(\lambda+r)} \, dt + e^{-(r+\lambda)T} (\bar{\psi} + v)$$

where $\bar{\psi} = \lim_{n \to \infty} \psi / n$. The first order condition for T gives:

$$\left(T - (r+\lambda) \frac{\bar{\psi}}{B\sigma^2} \right) \left(1 - e^{-(\lambda+r)T} \right) = (r+\lambda) \int_0^T t \, e^{-(\lambda+r)t} \, dt + e^{-(r+\lambda)T} (r+\lambda) \frac{\bar{\psi}}{B\sigma^2}$$

canceling terms, solving the integral, and taking $r \downarrow 0$ gives $\lambda T + e^{-\lambda T} - 1 = \lambda^2 \frac{\bar{\psi}}{B\sigma^2}$, which can we written as: $\bar{\psi} = \frac{B\sigma^2}{\lambda^2} \left[e^{-\lambda T} - 1 + \lambda T \right]$. From the dynamics of the price gaps as $n \to \infty$ we have that $\ell = \frac{\lambda}{N_a} = 1 - e^{-\lambda T}$, which combined with the fundamental lemma for price gaps $\sigma^2 = Var(\Delta p_i) \; N_a$. Using this in the expression for $\bar{\psi}$ we obtain:

$$\bar{\psi} \;\; = \;\; B \frac{Var(\Delta p_i) \, N_a}{\lambda^2} \left[-log(1-\ell) - \ell \right] = B \frac{Var(\Delta p_i)}{N_a} \left[\frac{-log(1-\ell) - \ell}{\ell^2} \right] .$$

□

Proof. (of Proposition 6) The proof has three parts. First we introduce a discrete time, discrete state version of the model, second we show the scaling of the time with N_a , and finally the homogeneity of degree one in terms of $Std[\Delta p_i]$ and δ.

Discrete Time Formulation. We start with discrete time version of the process for price gaps, with length of the time period Δ, which makes some of the arguments more accessible. Let N be a

$$N(t+\Delta) = \begin{cases} N(t) & \text{with probability } (1 - \lambda\Delta) \\ N(t) + 1 & \text{with probability } \lambda\Delta \end{cases} \tag{23}$$

Thus, as $\Delta \downarrow 0$ this process converges to a continuous time Poisson counter with instantaneous intensity rate λ per unit of time. Let $\bar{p}_i$ follow n drift-less random walks

$$\bar{p}_i(t+\Delta, p) = \begin{cases} \bar{p}_i(t,p) \; + \; \sigma \sqrt{\Delta} & \text{with probability } 1/2 \\ \bar{p}_i(t,p) \; - \; \sigma \sqrt{\Delta} & \text{with probability } 1/2 \end{cases} \tag{24}$$

where the initial condition satisfies:

$$\bar{p}_i(0) = p_i \text{ for } i = 1, .., n \; ,$$

and where the n random walks are independent of each other and of the Poisson counter. As $\Delta \downarrow 0$ the process for $\bar{p}$ converges to a Brownian motion whose changes have variance σ^2 per unit of time. We define the stopping time of the first price adjustment $\tau(p)$, conditional on the starting at price gap vector p at time zero, as:

$$\begin{aligned}\tau_1 &\equiv \min\{t = 0, \Delta, 2\Delta, \ldots : N(j\Delta + \Delta) - N(j\Delta) = 1\}, \\ \tau_2(p) &\equiv \min\left\{t = 0, \Delta, 2\Delta, \ldots : \sum_{i=1}^{n} (\bar{p}_i(j\Delta + \Delta, p))^2 \geq \bar{y}\right\} \text{ and} \\ \tau(p) &\equiv \min\{\tau_1 \,,\, \tau_2(p)\}.\end{aligned}$$

The function g is the density for the continuous time limit, i.e. the case where $\Delta \downarrow 0$. For small Δ, we can approximate the distribution of the *fraction* of firms with price gap vector p as the product of the density g and a correction to convert it into a probability, i.e a fraction. This gives:

$$g\left(p_1, ..., p_; n, \lambda/\sigma^2, \bar{y}\right) \left(\sigma\sqrt{\Delta}\right)^n$$

where the last term uses that in each dimension price gaps vary discretely in steps of size $\sigma\sqrt{\Delta}$. We can write the discrete time impulse response function as:

$$\mathcal{P}(t, \delta; \sigma, \lambda, \bar{y}, \Delta) \;=\; \Theta(\delta; \sigma, \lambda, \bar{y}, \Delta) + \sum_{s=\Delta}^{t} \theta(\delta, s; \sigma, \lambda, \bar{y}, , \Delta)\, \Delta\,,$$

In this expression we can, without loss of generality, restrict t to be an integer multiple of Δ. We have divided the expression for θ by Δ, and hence multiplied its contribution back by Δ in $\mathcal{P}$, so that it has the interpretation of the contribution per unit of time to the IRF of price changes at time t, i.e. it has the units of a density. Moreover, in this manner the term has a non-zero limit, and the expression in $\mathcal{P}$ converges to an integral. Thus we get the $\mathcal{P} = \lim \mathcal{P}(\Delta)$ as $\Delta \downarrow \infty$. The functions θ and Θ are given by:

$$\Theta(\delta; \sigma, \lambda, \bar{y}, \Delta) \equiv \sum_{||p(0)-\iota\delta|| \geq \bar{y}} \left(\delta - \frac{\sum_{j=0}^{n} p_j(0)}{n}\right) g\left(p(0); n, \frac{\lambda}{\sigma^2}, \bar{y}\right) \left(\sigma\sqrt{\Delta}\right)^n \text{, and}$$

$$\begin{aligned}&\theta(\delta, t; \sigma, \lambda, \bar{y}, \Delta) \equiv \\ &-\frac{1}{\Delta} \sum_{||p(0)-\iota\delta|| < \bar{y}} \mathbb{E}\left[\frac{\sum_{j=0}^{n} \bar{p}_j(t, p)}{n} \mathbf{1}_{\{\tau(p)=t\}} \;\middle|\; p = p(0) - \iota\delta\right] g\left(p(0); n, \frac{\lambda}{\sigma^2}, \bar{y}\right) \left(\sigma\sqrt{\Delta}\right)^n\end{aligned}$$

Time scaling of the IRF with N_a. For this (i) Note that if multiply the parameters σ^2 and λ by a constant $k > 0$, leaving $\bar{y}$ unaltered, then $N'_a = k\,N_a$, where primes are used to denote the values that correspond to the scaled parameters. This follows directly from the expression we derive for $N_a = 1/T(0)$ in Proposition 3. (ii) By Proposition 4 with these changes the distribution of price changes implied by $(\sigma^2, \lambda, \bar{y})$ is exactly the same as the one implied by $(k\sigma^2, k\lambda, \bar{y})$. (iii) we change notation and write $(\sigma^2, \lambda, \bar{y})$ instead of $(\lambda, \sigma^2, \psi/B)$ and omit n. We establish that

$$\mathcal{P}_n\left(\frac{t}{k}, \delta; k\sigma^2, k\lambda, \bar{y}\right) = \mathcal{P}_n\left(t, \delta; \sigma^2, \lambda, \bar{y}\right)$$

We will do so by establishing this proposition for the discrete time version of the IRF. Yet the result is immediate, since λ and σ^2 are the only two parameters which are rates per unit of time (the other parameters are n and $\bar{y}$), so by multiplying them by k we just scale time. The details can be found in the discrete time formulation, whose notation we develop below. We show that

$$\mathcal{P}(t, \delta; k\sigma^2, k\lambda, \bar{y}, \Delta/k) = \mathcal{P}(t/k, \delta; \sigma^2, \lambda, \bar{y}, \Delta) \tag{25}$$

We will do so by establishing this proposition for the discrete time version of the IRF. Let $\Delta' = \Delta/k$, $\sigma'^2 = \sigma^2 k$ and $\lambda' = \lambda k$. Note that, by construction $\sigma' \sqrt{\Delta'} = \sigma \sqrt{\Delta}$ and $\lambda'/(\sigma')^2 = \lambda/(\sigma)^2$. To establish this we first note that, for a given shock δ, Θ depends only on n, $\bar{y}$, $\sigma\sqrt{\Delta}$, and λ/σ^2. This is because the invariant density g and the scaling factor to convert it into probabilities depends only on those parameters. Second we show that

$$\sum_{s=\Delta/k}^{t/k} \frac{\Delta}{k} \theta\left(s, \delta; k\sigma^2, k\lambda, \bar{y}, \frac{\Delta}{k}\right) = \sum_{s=\Delta}^{t} \Delta\, \theta\left(s, \delta; \sigma, \lambda, \bar{y}, \Delta\right)$$

This follows because for each s and $p(0)$

$$\begin{aligned} & \mathbb{E}\left[\frac{\sum_{j=0}^{n} \bar{p}_j\left(s, p\right)}{n}\, \mathbf{1}_{\{\tau(p)=s\}} \;\middle|\; p = p(0) - \iota\delta\,;\, \sigma, \lambda, \Delta\right] \\ = \;& \mathbb{E}\left[\frac{\sum_{j=0}^{n} \bar{p}_j\left(\frac{s}{k}, p\right)}{n}\, \mathbf{1}_{\left\{\tau(p)=\frac{s}{k}\right\}} \;\middle|\; p = p(0) - \iota\delta\,;\, \sigma', \lambda', \Delta'\right] \end{aligned}$$

where we include the parameters $(\lambda, \sigma^2, \Delta)$ as argument of the expected values. This itself follows because, using equation (23) and equation (24) then the processes for $\{\bar{p}_i\}$ are the same in the original time and in the time time scales by k since the probabilities of the counter to go up $\lambda'\Delta' = \lambda\Delta$ and the steps of the symmetric random walks $\sigma'\sqrt{\Delta'} = \sigma\sqrt{\Delta}$ are the same in the original time and the time scaled by k. In particular we have that

$$\bar{p}_j\left(\frac{s}{k}, p; \lambda', \sigma'^2, \Delta'\right) \equiv \bar{p}_j\left(\frac{s}{k}, p; k\lambda, k\sigma^2, \frac{\Delta}{k}\right) = \bar{p}_j\left(s, p; \lambda, \sigma^2, \Delta\right) = \hat{p}$$

with exactly the same probabilities for each price gap $\hat{p} \in \mathbb{R}$ and each time $s \geq 0$. Also, repeating the arguments used for Θ, we have $g\left(p(0); n, \frac{\lambda}{\sigma^2}, \bar{y}\right) \left(\sigma\sqrt{\Delta}\right)^n = g\left(p(0); n, \frac{\lambda'}{\sigma'^2}, \bar{y}\right) \left(\sigma'\sqrt{\Delta'}\right)^n$. Thus, since equation (25) holds for all $\Delta > 0$, taking limits

$$\mathcal{P}\left(\frac{t}{k}, \delta; k\sigma^2, k\lambda, \bar{y}\right) = \lim_{\Delta\downarrow 0} \mathcal{P}\left(\frac{t}{k}, \delta; k\sigma^2, k\lambda, \bar{y}, \frac{\Delta}{k}\right) = \lim_{\Delta\downarrow 0} \mathcal{P}\left(t, \delta; \sigma^2, \lambda, \bar{y}, \Delta\right) = \mathcal{P}\left(t, \delta; \sigma^2, \lambda, \bar{y}\right)$$

Scaling of the IRF in the monetary shock with $Std[\Delta p_i]$. For this we use properties of the invariant distribution f, which are then inherited by g. In particular, we will compare the IRF with parameters $(\lambda, \sigma^2, \bar{y})$ with one with parameters $(\lambda', \sigma'^2, \bar{y})$ where $\lambda' = \lambda$, $\sigma'^2 = k\,\sigma^2$ and $\bar{y}' = k\,\bar{y}$. With this choice we have $N_a' = N_a$ and thus $\ell = \lambda'/N_a'$ since $\lambda\bar{y}/(n\sigma^2) = \lambda'\bar{y}'/(n\sigma'^2)$ (see Proposition 3). Then by Proposition 1 we have that the standard deviation of price changes scales up with k, i.e.: $Std[\Delta p_i]' = \sqrt{k}\, Std[\Delta p_i]$. The main idea is that the invariant distribution corresponding to the $'$ parameters is a radial expansion of the original, so that $\int_0^y f(x;\lambda,\sigma^2,\bar{y})dx = \int_0^{yk} f(x;\lambda',\sigma'^2,\bar{y}')dx$ and thus $f(y,\lambda,\sigma^2,\bar{y}) = kf(yk,\lambda',\sigma'^2,\bar{y}')$. Indeed using Lemma 2 we have:

$$f\left(y;\frac{\lambda}{\sigma^2},\bar{y}\right) = k\, f\left(yk;\frac{\lambda}{k\sigma^2},k\bar{y}\right) \equiv k\, f\left(yk;\frac{\lambda'}{\sigma'^2},\bar{y}'\right) \ . \tag{26}$$

Thus we have:

$$\begin{aligned}
g\left(p_1,...,p_;n,\frac{\lambda}{\sigma^2},\bar{y}\right) &= f\left(p_1^2+\cdots+p_n^2;n,\frac{\lambda}{\sigma^2},\bar{y}\right)\frac{\Gamma\left(n/2\right)}{2\,\pi^{n/2}\ \left(p_1^2+\cdots+p_n^2\right)^{(n-2)/2}} = \\
&= kf\left(k(p_1^2+\cdots+p_n^2);n,\frac{\lambda'}{\sigma'^2},\bar{y}'\right)\frac{\Gamma\left(n/2\right)k^{(n-1)/2}}{2\,\pi^{n/2}\ \left(k(p_1^2+\cdots+p_n^2)\right)^{(n-2)/2}} \\
&= g\left(\sqrt{k}(p_1,...,p_n);n,\frac{\lambda'}{\sigma'^2},\bar{y}'\right)k^{(n-2)/2}\,k
\end{aligned}$$

Using this for the discrete time formulation we have:

$$\begin{aligned}
g\left(p;n,\frac{\lambda}{\sigma^2},\bar{y}\right)\left(\sigma\sqrt{\Delta}\right)^n &= g\left(\sqrt{k}\,p;n,\frac{\lambda'}{\sigma'^2},\bar{y}'\right)\left(\sigma'\sqrt{\Delta}\right)^n k^{(n-2)/2}\,k\,k^{-n/2} \\
&= g\left(\sqrt{k}\,p;n,\frac{\lambda'}{\sigma'^2},\bar{y}'\right)\left(\sigma'\sqrt{\Delta}\right)^n
\end{aligned}$$

Note that $\{||p(0)-\iota\delta|| \geq \bar{y}\} = \{||\sqrt{k}\,p(0)-\iota\sqrt{k}\,\delta|| \geq \sqrt{k}\,\bar{y}\} = \{||\sqrt{k}\,p(0)-\iota\delta'|| \geq \bar{y}'\}$. Also

$$\left(\delta - \frac{\sum_{j=0}^n p_j(0)}{n}\right)\sqrt{k} = \left(\delta' - \frac{\sum_{j=0}^n \sqrt{k}\,p_j(0)}{n}\right)$$

Thus

$$\begin{aligned}
&\sqrt{k}\sum_{||p(0)-\iota\delta||\geq\bar{y}}\left(\delta - \frac{\sum_{j=0}^n p_j(0)}{n}\right)g\left(p(0);n,\frac{\lambda}{\sigma^2},\bar{y}\right)\left(\sigma\sqrt{\Delta}\right)^n \\
= &\sum_{||\sqrt{k}p(0)-\iota\delta'||\geq\bar{y}'}\left(\delta' - \frac{\sum_{j=0}^n \sqrt{k}p_j(0)}{n}\right)g\left(\sqrt{k}\,p(0);n,\frac{\lambda'}{\sigma'^2},\bar{y}'\right)\left(\sigma'\sqrt{\Delta}\right)^n
\end{aligned}$$

Using the definition of $\Theta(\cdot,\Delta)$:

$$\sqrt{k}\,\Theta(\delta;\sigma,\lambda,\bar{y},\Delta) = \Theta\left(\sqrt{k}\,\delta;k\sigma^2,\lambda,k\bar{y},\Delta\right) \equiv \Theta\left(\delta';\sigma'^2,\lambda',\bar{y}'\Delta\right) \ .$$

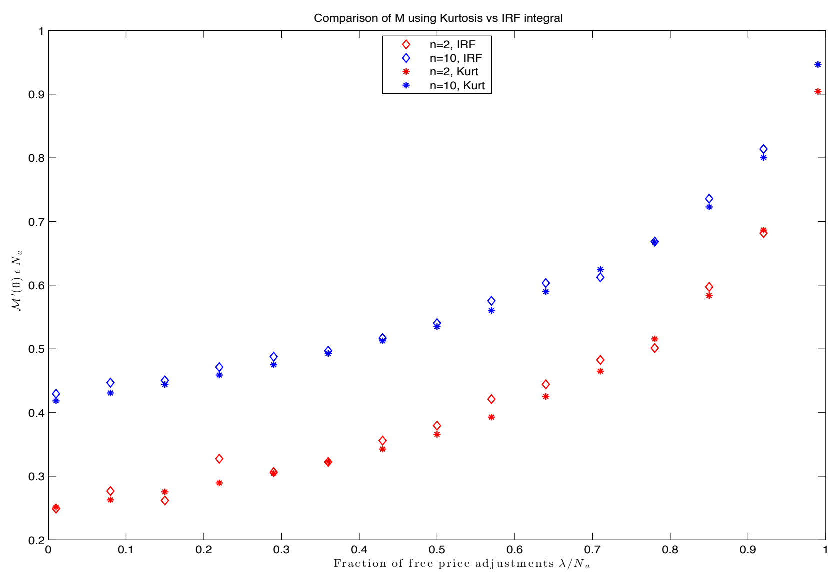
Since this holds for all Δ, by taking limits as $\Delta \downarrow 0$, we have shown the desired result for Θ. The result for θ follows the steps for g. We set $\Delta' = \Delta$ and note that for all $p(0) \in \mathbb{R}^n$, scaling factor $k > 0$ and time horizon $s > 0$:

$$
\begin{aligned}
&\sqrt{k}\ \mathbb{E}\left[\frac{\sum_{j=0}^{n}\bar{p}_j\left(s,p\right)}{n}\ \mathbf{1}_{\{\tau(p)=s\}}\ \Big|\ p = p(0) - \iota\delta\,;\,\sigma,\lambda,\Delta\right] \\
= &\ \ \mathbb{E}\left[\frac{\sum_{j=0}^{n}\bar{p}_j\left(s,p\right)}{n}\ \mathbf{1}_{\{\tau(p)=s\}}\ \Big|\ p = \sqrt{k}\,p(0) - \iota\delta'\,;\,\sigma',\lambda',\Delta\right].
\end{aligned}
$$

This follows because $\lambda' = \lambda$ and $\sigma'\sqrt{\Delta'} = \sqrt{k}\,\sigma\,\sqrt{\Delta}$, thus the each $p \in \mathbb{R}^n$ the paths $\sqrt{k}\,\bar{p}(s,p;\sigma,\lambda) = \bar{p}(s,\sqrt{k}p;\sigma',\lambda')$ occur with the same probabilities. $\square$

Proof. (of Proposition 7). For the cases of $n = 1$ and $n = \infty$, in Appendix H we give closed form expressions for the kurtosis $kurt\,(\Delta p_i)$ and for the cumulated output effect, $\mathcal{M}(\delta)_{n,\ell}$: it is immediate to verify that they are proportional to each other, as stated in the proposition.

Figure 7: Cumulated output effect: Kurtosis vs Area under IRF



For $1 < n < \infty$ we do not have a closed form expression for $\mathcal{M}_{n,\ell}(\delta)$, and only a semi-analytical expression for $kurt\,(\Delta p_i)$. We stress that, as emphasized by Proposition 6, both the Kurtosis and $\mathcal{M}_{n,\ell}(\delta)/(\delta\epsilon N_a) \cong \mathcal{M}'_{n,\ell}(0)$ depend *only* on 2 parameters: n and ℓ. Therefore, for a given n, the verification of Proposition 7 requires to check that Kurtosis and $\mathcal{M}'_{n,\ell}(0)$ are the same at all values of $\ell \in (0,1)$. We outline a procedure for doing this verification numerically. The proof of Proposition 6 defines the discrete time analog of $\mathcal{P}(\delta)_{n,\ell}$ in detail, which allows for the numerical computation of $\mathcal{M}_{n,\ell}(\delta)$ by numerical integration. Verifying the proposition amounts to numerically checking, for a given value of n that the two expressions are proportional. The matlab code to verify the statement, up to numerical precision, is posted on our website. An example of the accuracy of the results for $n = 2$ and $n = 10$ is plotted in Figure 7. $\square$

ADDITIONAL MATERIAL (NOT FOR PUBLICATION)

Small and large price changes in menu cost models

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B Related literature on stochastic menu costs

Models with random cost of adjustment have been introduced by Caballero and Engel (1999) and Dotsey, King, and Wolman (1999). Caballero and Engel (1999) study and solve numerically a model of investment with random fixed cost of adjustement. Two outcomes are (i) that the decision rule has the form of a "generalized (S,s) rule", thus time-varying inaction thresholds, and (ii) that higher moments of the cross-sectional distribution of firm disequilibria can predict aggregate investment. Dotsey, King, and Wolman (1999) introduce random menu cost in a price-setting context, to develop a tractable general equilibrium model of state-dependent pricing. While at the individual level, adjustment is discrete, the introduction of random menu costs makes the fraction of adjusting firms become a relevant state variable which reacts smoothly to shocks. The model can then be solved with standard linearization techniques, using the property that it is sufficient to keep track of vintages of firms each characterized by the same reset dates.[39] Dotsey, King, and Wolman (1999) and Dotsey and King (2005) use the model to investigate how the response to monetary policy shocks under state–dependent pricing differ from that time–dependent pricing.

Recently, a series of papers have used random menu cost models with the explicit aim of fitting of micro data on price changes.[40] Dotsey, King, and Wolman (2009) follow up on Dotsey, King, and Wolman (1999) by introducing idiosyncratic shocks and calibrating the model using inter alia the distribution of micro economic price changes in the US. Caballero and Engel (2007) apply the generalized hazard approach of Caballero and Engel (1999) to price dynamics and illustrate how introducing random free opportunity of price changes alters the response of the economy to a monetary shock. Midrigan (2011) show that economies of scales in price setting for a multiproduct firm, and random menu costs, are alternative mecanisms that generate small price changes at the individual level. He concludes that under either economies of scales in price setting, or random menu costs, monetary policy have more persistent effect than in the Golosov and Lucas (2007) menu cost model. Burstein and Hellwig (2006) reach the same conclusion when adding random menu cost in a model with pricing complementarity. Nakamura and Steinsson (2010) also examine, in a multisector menu cost model, to which extend monetary non-neutrality is increased in a variant of the model in which the menu cost can randomly receive a low or high value. Woodford (2009) develops a model of price-setting under information capacity constraint. Optimal policy gives rise to randomisation of the price review decision. Costain and Nakov (2011, 2012) develop a model in which the probability of adjustment is a function of the value of adjustment for firms. Both in Woodford (2009) and Costain and Nakov (2011, 2012), the model is calibrated using moments of the distribution of price changes from micro data, and the obtained decision rule is observationally equivalent to that derived under a random menu cost model. Overall, two common features of this series of recent models is that they are solved using numerical techniques, and they obtain that under random menu cost the degree of monetary policy non-neutrality is to some extent larger than in the fixed menu cost model of Golosov and

[39] The model is set-up under the assumptions of no idiosyncratic shocks and i.i.d. random cost, so all firms that reset price set the same price. To have a finite number of vintages, the model requires positive steady state inflation.

[40] Previous to the recent research, Willis (2000) had estimated a partial equilibrium model stochastic menu cost model on magazine data

Lucas (2007).

The present paper is related to this recent literature. A distinctive feature is that results are derived analytically, and the way the impact of monetary policy shock depends on "deep" parameters is studied in a systematic way. The model with random menu cost is also extended to incorporate economies of scales in price adjustment.

C Data Appendix

C.1 Details on data treatment and further sectoral statistics

Some additional features of our data treatment are as follows.

Dealing with product replacement. The dataset contains flags for product replacement as well as imputed prices which we use as follows to design our dataset. First, we discarded observations with item substitution, as item substitution may result into spurious values for price changes, if quality adjustment is not accounted for or imperfectly measured (Berardi, Gautier, and Le Bihan (2013) investigate the inclusion of information on item substitutions). Second, we replaced any "imputed price" in the dataset, by the previous price of the same item in the same outlet present in the data, i.e. a carry–forward procedure. In the source dataset imputed prices are introduced by the INSEE when prices are missing.[41] Imputed prices are constructed either using the carry–forward procedure, or imputing the average price change of similar goods observed in the close area. The latter procedure makes sense from the aggregate CPI point of view but is obviously ill-suited for characterizing price change at the individual level. We used the flag for imputed prices to locate and replace them by carry-forward prices. This procedure amounts to discarding imputed prices when computing the distribution of (non-zero) price changes.

Computing price changes and dealing with outliers. Price changes were computed as 100 times the log-difference in prices per unit. We compute a consistent price per unit by, when relevant, dividing prices by the indicator of quantity sold (package size). We removed outliers, which in our baseline analysis we define as price changes smaller in absolute value than 0.1 percent, or larger in absolute value than $100 \cdot \log(10/3)$. These thresholds are set as a first crude ways to deal with measurement errors. Some robustness checks are presented in Table 7. The upper threshold for outliers is set with sales in mind, as we informally observe that price rebates as large as 70% are sometime advertised in sales periods. Our threshold allows for a price to decrease by up to 70% and subsequently return to its former level without discarding the observation. Price changes larger than this threshold are discarded as being outliers.[42]

Identifying sales. The flag for sale allows to identify sales. Two kinds of sales-promotion discounts, that have a different status, exist in France: seasonal sales or temporary discounts. Seasonal sales ('soldes') are subject to administrative restrictions: the time period (twice a year) is decided by local authorities and price posting is subject to precise regulations. Temporary discounts are not subject to such restrictions but sales below cost are prohibited by

[41] Prices may be missing because of stock-outs, closed outlet due e.g. to holidays or seasonality in product availability, for instance.

[42] An example of outlier is the fee for parking in the street, which is free in some cities in summer.

commercial law. By contrast, selling below cost is allowed in the case of seasonal sales. On the sample period, seasonal sales are observed only in some specific categories of goods (mainly clothes). The proportion of price quotes that are flagged as seasonal sales is 0.76% and the proportion of temporary discounts amounts to 1.92%.

Main facts at sectoral level. The different sectors in the CPI have very different pricing patterns, as well documented in recent research. The purpose of this appendix section is to illustrate that the peakedness of the price change distribution is a fact observed in all sectors. Table 6 documents pricing patterns fact using a breakdown 6 into broad economic sectors.[43] As previous research, we observe many sectoral specificities: prices change less often and rarely decrease in services; the size of price changes is smaller in services; energy prices change frequently and by small amounts; reflecting sales, the variance of price change is huge in clothes. However, noticeably a large kurtosis is observed in all sectors, one exception being clothes for which kurtosis (2.09) is lower than that of the Gaussian distribution. The fraction of small price changes, using one fourth of mean absolute price change as a threshold, ranges between 8% and 27% for all categories other than energy. Using a sector and type of good partition, further documents that this fact is consistently observed at higher levels of disaggregation.

Table 6: Results by type of goods

Good type	**Freq**	**Avg** $\lvert\Delta p\rvert$	**Std** $\lvert\Delta p\rvert$	**Kurt** (Δp)	**Frac25**
Food	19.38	9.18	12.31	10.78	29.26
Durable goods	15.16	14.73	13.57	5.99	18.07
Clothing	11.00	42.48	24.71	2.16	10.21
Other manufactured goods	11.43	10.39	14.34	9.36	34.02
Energy	77.00	3.79	3.10	6.90	12.13
Services	6.53	7.80	10.29	17.58	21.29

Source is INSEE, monthly price records from French CPI, data from 2003:4 to 2011:4. Coverage is aroud 65% of CPI weight since rents, and prices of fresh food and centrally collected items (e.g. electricity, train and airplane tickets) are not included in the dataset. Freq. denotes monthly frequency of price change in percent. Size of price change Δp are the first-difference in the logarithm of price per unit, expressed in percent. Avg is average, Std standard devation, Frac25 the share of absolute price change that are inferior to 0.25 Avg[$\lvert\Delta p\rvert$], Kurt denotes kurtosis. Observations with imputed prices or quality change are discarded. Moments are computed aggregating all prices changes using CPI weigths at the product level.

C.2 Small price changes and measurement error

This appendix examines to what extent the arguments of Eichenbaum et al. (2013) apply to our data and investigates the robustness of our findings to various criteria for trimming the

[43]The breakdown we use (food; durable goods; clothing & textile; other manufactured goods ; energy;services) is one we deem the most meaningful to capture price-setting idiosyncracies.

Table 7: Robustness to trimming

Type of trimming	Flag	Freq.	Avg($\lvert\Delta p\rvert$)	Std[$\lvert\Delta p\rvert$]	Frac25	Kurt[Δp_i]	Kurt[z]
$\lvert\Delta p_i\rvert \leq 100 \cdot \log(2)$	1	17.10	8.51	12.00	28.93	10.23	7.35
Exc. flagged sales	2	14.82	5.05	5.90	18.77	13.59	8.60
$\lvert\Delta p_i\rvert \leq 100 \cdot \log(10/3)$	3	17.21	9.12	13.79	30.33	12.92	9.04
$0.1 \leq \lvert\Delta p_i\rvert \leq 100 \cdot \log(2)$	4	16.98	8.59	12.03	28.48	10.14	7.21
$0.5 \leq \lvert\Delta p_i\rvert \leq 100 \cdot \log(2)$	5	16.56	8.84	12.12	27.06	9.84	6.86
$0.1 \leq \lvert\Delta p_i\rvert \leq 100 \cdot \log(10/3)$ & ex.sales	6	14.70	5.15	6.23	18.21	20.86	10.40
$0.1 \leq \lvert\Delta p\rvert \leq 100 \cdot \log(10/3)$	8	17.09	9.19	13.82	29.91	12.81	8.89
$1 \leq \lvert\Delta p_i\rvert \leq 100 \cdot \log(2)$	11	15.27	9.66	12.44	22.46	8.94	6.33

(Table, continued) Moments of standardized price change

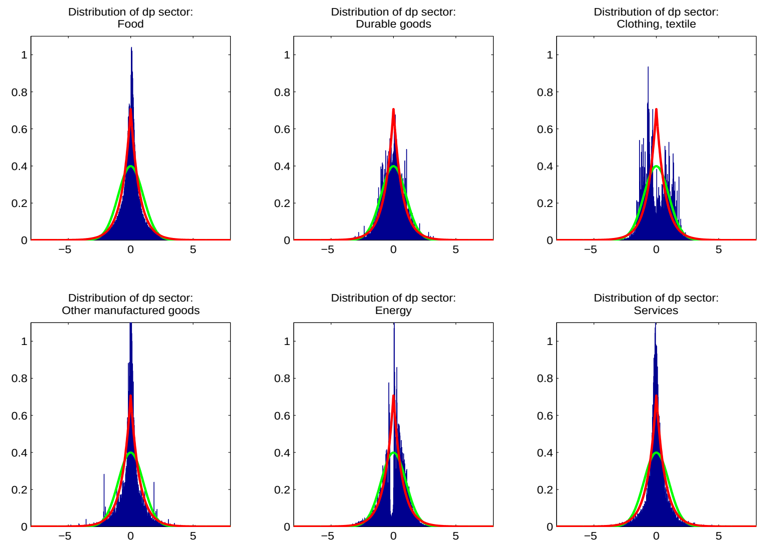
Type of trimming	Flag	Frac($< 0.25m$)	Frac($< 0.5m$)	Frac($> 2m$)	Frac($> 4m$)
$\lvert\Delta p_i\rvert \leq 100 \cdot \log(2)$	1	39.29	22.01	13.10	1.75
Exc. flagged sales	2	38.59	20.62	12.58	1.97
$\lvert\Delta p_i\rvert \leq 100 \cdot \log(10/3)$	3	39.55	22.25	12.95	1.82
$0.1 \leq \lvert\Delta p_i\rvert \leq 100 \cdot \log(2)$	4	39.10	21.90	13.07	1.72
$0.5 \leq \lvert\Delta p_i\rvert \leq 100 \cdot \log(2)$	5	38.36	20.91	12.85	1.61
$0.1 \leq \lvert\Delta p_i\rvert \leq 100 \cdot \log(10/3)$ & ex.sales	6	38.55	20.67	12.51	1.96
$0.1 \leq \lvert\Delta p\rvert \leq 100 \cdot \log(10/3)$	8	39.31	22.18	12.91	1.79
$1 \leq \lvert\Delta p_i\rvert \leq 100 \cdot \log(2)$	11	35.61	17.74	12.09	1.29

Source is INSEE, monthly price records from French CPI, data from 2003:4 to 2011:4. Coverage is aroud 65% of CPI weight since rents, and prices of fresh food and centrally collected items (e.g. electricity, train and airplane tickets) are not included in the dataset. Freq. denotes monthly frequency of price change in percent. Size of price change Δp are the first-difference in the logarithm of price per unit, expressed in percent. Avg is average, Std standard devation, Frac25 the share of absolute price change that are inferior to 0.25 Avg[$\lvert\Delta p\rvert$], Kurt denotes kurtosis. Kurt[z] denotes kurtosis of the distribution of standardized price changes. Standardized price changes are computed at the category of good * type of outlet level. Observations with imputed prices or quality change are discarded. Moments are computed aggregating all prices changes using CPI weigths at the product level. Each row describes a sub-sample constructed applying the filter described by the column "type of trimming". "Ex. sales" exclude observations flagged as sales by the INSEE data collectors. The subsample with flag code 8 is taken as the baseline in the main text of the paper.

data. Measurement errors may arise for several reasons. Eichenbaum, Jaimovich, and Rebelo (2011) and Eichenbaum et al. (2013) articulate two concerns about the small price change. First they notice that in scanner data studies the price level of an item is typically computed as the ratio of recorded weekly revenues to quantity sold. To the extent that there are temporary or individual specific discounts (say coupons), this will generate spurious small price changes.[44] Moreover Eichenbaum et al. (2013) highlight a related problem for some CPI items: they spot 27 items (named ELIS in the BLS terminology) that are problematic because these prices are typically computed as a Unit Value Index (a ratio of expenditure

[44] Notice that in principle CPI data are immune from this type of measurement error, as these data are direct transaction prices observed by a field agent. Indeed, in the instance of a temporary discount, the CPI dataset will record either no price change, or the large price change of observed during the discount, if the field agent happens to be collecting data during the temporary discount. Further, the protocol of data collection requires that the field agent records the price faced by a regular customer, not benefiting from individual-specific discounts.

Figure 8: Distribution of standardized Price Adjustments by group of goods



The figures uses the elementary CPI data from France 2003-2011 (see the text).

to quantity purchased), or they are not consistently recorded in the same outlet, or they are the price of a bundle of goods (for instance the sum of airplane fare and airport tax). We were able to match these items with their counterparts in our French dataset. Out of the 27 problematic items 15 are not present in our data because in the French CPI those items are not recorded by a field agent but are centrally collected (thus not made available in the subset of CPI we have access to).[45] Concerning the 12 remaining items virtually no price record in the French CPI is computed as a Unit Value Index, which is hypothesized by Eichenbaum et al. (2013) as a major source of small price changes. Inspecting the patterns of price changes over these 12 potentially "problematic" items in our dataset shows that the amount of small price changes is not significantly different from the one detected over the rest of our sample. One exception is the price of "Residential water" where it can be suspected that many small variations in local taxes occur.[46]

A second investigation on measurement error was developed by varying the upper and lower thresholds of small and large price changes used to define outliers. Results are displayed in Table 7 of the Appendix. In each of the variants considered in Table 7, both kurtosis and the fraction of small price changes remain large. The lowest level of kurtosis obtains when we use the most stringent thresholds for outliers.

[45]These items are Hospital room in-patient; Hospital in-patient services other than room ; Electricity; Utility natural gas service; Telephone services, local charges ; Interstate telephone services ; Community antenna or cable TV ; Cigarettes; Garbage and trash collection; Airline fares; New cars; New trucks; Ship fares; Prescription drugs and medical supplies; Automobile insurance.

[46]Otherwise, on the bulk of consumption items, there are no local taxes in France, and the main, nation-wide, rate of the Value Added Tax rate did not move over the sample period.

D Details of the solution for the model with $n = 1$

Integrating the Bellman equation gives the following value function

$$v(p) = \frac{Bp^2 + \lambda v(0)}{\lambda + r} + \frac{B\sigma^2}{(\lambda + r)^2} + C\left(e^{p\sqrt{\frac{2(\lambda+r)}{\sigma^2}}} + e^{-p\sqrt{\frac{2(\lambda+r)}{\sigma^2}}} \right)$$

where we already used that $v(p) = v(-p)$. Notice that the value function has a minimum (and zero derivative) at $p = 0$, which is the optimal return point. The constant C and the threshold value $\bar{p}$ are the values that solve the 2 equation system given by the value matching condition and the smooth pasting conditions.

The expected time to adjustment, $T(p)$ obeys the differential equation $\lambda T(p) = 1 + \frac{\sigma^2}{2} T''(p)$ with boundary condition $T(\bar{p}) = 0$. Given the symmetry of the law of motion for p, the function is symmetric, i.e. $T(p) = T(-p)$. Integrating gives $T(p) = \frac{1}{\lambda}\left(1 - \frac{e^{\sqrt{\frac{2\lambda}{\sigma^2}}p} + e^{-\sqrt{\frac{2\lambda}{\sigma^2}}p}}{e^{\sqrt{\frac{2\lambda}{\sigma^2}}\bar{p}} + e^{-\sqrt{\frac{2\lambda}{\sigma^2}}\bar{p}}}\right)$.

The distribution of price gaps $h(p)$ satisfies the Kolmogorov forward equation $0 = -\frac{2\lambda}{\sigma^2} h(p) + h''(p) \quad \text{for} \quad 0 < |p| \leq \bar{p}$. The density is symmetric, $h(p) = h(-p)$, and satisfies the boundary conditions: $h(\bar{p}) = 0$ and it integrates to one i.e. $2\int_0^{\bar{p}} h(p)\, dp = 1$ where we used that it is symmetric.[47]

E Note on Solutions of value function V, expected time to adjust $\mathcal{T}$ and invariant density of the squared price gap f.

First we state a proposition which gives an explicit closed form solution to the value function $v(y)$ in the inaction region, i.e. for $y \in (0, \bar{y})$ subject to $v(0) < \infty$. The solution is parameterized by $\beta_0 = v(0)$.

Proposition 8 *Let $\sigma > 0$. The ODE in equation (3) is solved by the analytical function: $v(y) = \sum_{i=0}^{\infty} \beta_i \, y^i$, for $y \in [0, \bar{y}]$ where, for any β_0, the coefficients $\{\beta_i\}$ solve: $\beta_0 = \frac{n\sigma^2}{r}\beta_1$, $\beta_2 = \frac{(r+\lambda)\beta_1 - B}{2\sigma^2(n+2)}$, $\beta_{i+1} = \frac{r+\lambda}{(i+1)\sigma^2\ (n+2i)}\ \beta_i$ for $i \geq 2$.*

The function described in this proposition allows to fully characterize the solution of the firm's problem. One can use it to evaluate the two boundary conditions described above, value matching and smooth pasting, and define a system of two equations in two unknowns, namely β_0 and $\bar{y}$.

The alert reader may have noticed that to solve for the invariant density f we have followed a standard procedure, i.e. set a 2nd order ordinary linear difference equation (the Kolmogorov forward equation) and find its solutions in terms of two constant, and using two boundary conditions to find the value of the constants. Instead to solve for V and $\mathcal{T}$ we

[47] The first boundary can be derived as the limit of the discrete time, discrete state, low of motion where each period is of length Δ and where p increases or decreases with probability 1/2, so that $h(p) = \frac{1}{2}h(p + \Delta) + \frac{1}{2}h(p - \Delta)$. At the boundary $\bar{p}$ this law of motion is $h(\bar{p}) = \frac{1}{2}h(\bar{p} - \Delta)$, which shows that $h(\bar{p}) \downarrow 0$ as $\Delta \downarrow 0$.

have followed a different approach, we guess an infinite expansion around $y = 0$ and compute its coefficients. Additionally, it may have looked that we did not provide enough boundary conditions to be able to solve for $\mathcal{T}$ and V. For instance, for $\mathcal{T}$ we gave only one equation as boundary conditions, namely $\mathcal{T}(\bar{y}) = 0$. Here we explain that we could have followed the more standard route, which required an analysis of the behavior close to the $y = 0$ boundary, to set one constant to zero and also would have produced a less informative result, i.e. one in terms of modified Bessel functions. Nevertheless we include it here for completeness.

Note that $V(y)$, $\mathcal{T}(y)$ and $f(y)$ are solutions to a linear ODE on y whose homogeneous component, say $q(\cdot)$, solves :

$$y\, q''(y) + a\, q'(y) + b\, q(y) \;=\; 0 \tag{27}$$

for $y \in [0, \bar{y}]$, for (different) constants a and b, with different particular solution, and different boundary conditions. The general solution of the homogeneous equation (27) is given by:

$$q(y) = |b\, y|^{(1-a)/2} \left[C_1 I_\nu \left(2\sqrt{|b\, y|} \right) + C_2 K_\nu \left(2\sqrt{|b\, y|} \right) \right] \tag{28}$$

provided that $b\, y < 0$, i..e. that $b < 0$, where C_1 and C_2 are arbitrary constants, $\nu = |1-a|$ and where I_ν and K_v are the modified Bessel functions of the first and second kind respectively. The values of $b = -\lambda/(2\sigma^2)$ in the three cases. The value of $a = n/2$ for $\mathcal{T}$ and for V, which are the same Kolmogorov backward equation, and $a = -(n/2 - 2)$ for f, which is the Kolmogorov forward equation.

It is important to notice the behavior of $I_\nu(z)$ and $K_\nu(z)$ for values of $0 < z$ but very close to zero. We have:

$$I_\nu \backsim \frac{1}{\Gamma(\nu+1)} \left(\frac{z}{2} \right)^\nu \tag{29}$$

and

$$K_\nu \backsim \begin{cases} \frac{\Gamma(\nu+1)}{2} \left(\frac{2}{z} \right)^\nu & \text{if } \nu > 0 \\ -\log(z/2) - \gamma & \text{if } \nu = 0 \end{cases} \tag{30}$$

We thus have that each of the solution will behave as:

$$\begin{aligned} I_{|1-a|} \left(y^{1/2} \right) \; y^{(1-a)/2} \quad &\backsim \quad \frac{1}{\Gamma(|1-a|+1)} \left(\frac{y^{1/2}}{2} \right)^{|1-a|} \quad y^{(1-a)/2} \\ &= \quad \frac{1}{\Gamma(|1-a|+1)} \left(\frac{1}{2} \right)^{|1-a|} \quad y^{(1-a)/2 + |1-a|/2} \end{aligned}$$

So if $1 - a = -|1-a|$, i.e. if $1 - a \leq 0$, the value of this product is finite at $y \downarrow 0$. Otherwise it diverges to ∞. Likewise for $\nu = |1-a| > 0$:

$$\begin{aligned} K_{|1-a|} \left(y^{1/2} \right) \; y^{(1-a)/2} \quad &\backsim \quad \frac{\Gamma(|1-a|+1)}{2} \left(\frac{2}{y^{1/2}} \right)^{|1-a|} \quad y^{(1-a)/2} \\ &= \quad \frac{\Gamma(|1-a|+1)}{2} \left(\frac{2}{1} \right)^{|1-a|} \quad y^{(1-a)/2 - |1-a|/2} \end{aligned}$$

So if $1-a = |1-a|$, i.e. if $1-a \geq 0$, the value of this product is finite at $y \downarrow 0$. Otherwise it diverges to ∞. The case of $\nu = 0$ i.e. $a = 1$ is special, but $K_0(z)$ also diverges and $I_0(z)$ converges to a non-zero constant as $z \downarrow 0$.

Note that $V(0)$ and $\mathcal{T}(0)$ are both finite. For these two cases the Kolmogorov backward equation has $a = n/2$ so $1-a \geq 0$ iff $n \geq 2$. In these cases we have that C_2, the constant associated with K_ν must be zero. We can use the constant C_1 to impose the boundary condition $\mathcal{T}(\bar{y}) = 0$ for $\mathcal{T}$ and to have a one dimensional representation of V in the range of inaction given $\bar{y}$. Then we can use smooth pasting and value matching, i.e. two boundary conditions, to find the constants C_1 and $\bar{y}$.

Note that for f we don't require that $f(0)$ be zero, since the density at zero gap can be infinite if the y mean reverts to zero fast enough. Thus in this case we will, in general, have both constants be non-zero.

F Fat-tailed shocks

This appendix compares the baseline multi-product model with random free adjustment opportunities with an otherwise "equivalent" multi-product model with fat-tailed shocks to costs. We present three propositions which show that:

1. If the fat-tailed shocks are sufficiently large, the threshold for adjustment is the same as in the model with random free adjustment opportunities.
2. The distribution of price changes with fat-tailed shock is different, since it includes the large shocks, and thus it contributes to kurtosis by mostly adding large price changes.
3. Since the model has more parameters, mainly the distribution of the fat-tailed shocks, it can capture more behavior, or putt it differently it is hard to identify the parameters with the same observations.

Set-up with fat-tailed shocks. Assume that the price gap for each product evolve as follows:

$$dp_i(t) = \sigma \, dW_i(t) \, + \, \xi_i(t) \, dN(t) \quad \text{for } i = 1, ..., n$$

where $N(t)$ is the counter of a Poisson process with intensity $\lambda \geq 0$. When $dN(t) = 1$, the price gap has a change of size ξ_i. The vector $\xi = (\xi_1, ..., \xi_n)$, is drawn from a joint distribution with cdf Γ, assumed to be symmetric around zero. Furthermore we assume that the marginal distribution of each of the coordinates of the vector ξ are identical. Each of the W_i are standard BMs, independent across products i. The realization of the Poisson counter N is common across the n products. This stylized set-up is meant to capture a generalization (to any $n \geq 1$) of the fat-tailed shocks in Midrigan's (2011) model.

Value function. The state of the problem is $p = (p_1, ..., p_n)$. For all $p \in \mathbb{R}^n$ the value function must solve the variational inequalities:

$$\begin{aligned} r\, v\,(p_1, ..., p_n) \leq\ & B\left[\sum_{i=1}^{n} p_i^2\right] + \sum_{i=1}^{n} \frac{\sigma^2}{2}\, v_{ii}(p_1, ..., p_n) & (31) \\ & + \lambda \int \cdots \int \min\left\{0\,,\ \psi + v(0) - v\,(p_1 + \xi_1, ..., p_n + \xi_n)\right\} d\Gamma\,(\xi_1, \cdots, \xi_n)\ , \\ v\,(p_1, ..., p_n) \leq\ & v(0) + \psi\ . & (32) \end{aligned}$$

with at least one of the two inequalities holding as equality at each vector $p \in \mathbb{R}^n$.

The variational inequality (31)-(32) has the advantage that it does not presume in the form of the control and of the inaction region, which is a delicate issue for a stopping time with fixed cost and jumps in the state. Nevertheless a function that solves this variational inequality must be the solution of the problem.

This problem has more parameters. As in the problem with free adjustment opportunities we have one integer and four positive scalars: n, B, σ^2, r, ψ. The new element on this formulation is the function $\Gamma : \mathbb{R}^n \to [0, 1]$ for the distribution of the fat-tailed shocks.

No small fat-tailed shock. We assume that the shocks are bounded below. We consider distributions where:

$$0 < \underline{\xi} \equiv \inf ||\xi||\ :\ (\xi_1, ..., \xi_n) \in \text{supp } \Gamma \text{ and } \xi \neq 0\ , \tag{33}$$

so that the minimization is on the support of Γ for which not all the coordinates are equal to zero. The assumption that $\underline{\xi} > 0$ is very natural for the case of one product to capture fat-tails. It is not clear what is the most natural generalization for multi-product case of $n > 1$. Note that the assumption so far allows that the components of ξ_i to be independent or not. In the case of independence, we can either assume that each coordinate of ξ_i as a strictly positive support, or otherwise assume that there is a mass point at zero and that the remaining of the support is strictly positive. In the latter case, the probability of the event in which all the coordinates are simultaneously zero, can be ignored but suitable rescaling λ.[48]

Simple threshold policies. We say that a firm follows a simple threshold policy if there is a threshold $\bar{y} > 0$ for which the firm adjust its price the first time that $||p||^2 \geq \bar{y}$ reverting its price gap to 0 in all components. Moreover, we require that the firm change prices every time a fat-tailed shock arrives.

Note that a possibility is that for the general case of a stopping time with fixed cost and jumps in the state there is the possibility that the optimal policy will be given by a inaction region made of the union of disconnected sets, see Alvarez and Lippi (2013a) for a discussion and references on the applied math literature. The next lemma gives an intermediate step

[48] Another case is the one in which only one product is subject to a large shock, a case we refer to a the isolated shock case. This is equivalent to make the Poisson shock independent for the arrival of the large changes to be independent across the n products.

to be used for the characterization of simple threshold policies in the proposition below. The lemma finds a lower bound on the support of the shocks, relative to the threshold for the norm of the state, so that the shocks will always take the post-shock state outside the region where it is smaller than the threshold.

Lemma 3 *Let p be a vector satisfying $||p||^2 \leq \bar{y}$, and consider the size of the square norm of the state after the occurrence of a large shock $||p+\xi||^2$. Then*

$$||p+\xi||^2 \geq \bar{y} \text{ for all } \xi \text{ for which } ||\xi|| \geq \left(1+\sqrt{2}\right)\sqrt{\bar{y}}\,.$$

The next proposition gives a straightforward way to characterize simple threshold policies. It says that if the fat-tailed shocks are sufficiently large so that they always trigger an adjustment, then one can use the same formulas than for the model with free adjustment opportunities, for which we have a complete characterization of $\bar{y}$.

Proposition 9 *Optimality of simple threshold policies for any $n \geq 1$. Let $\bar{y} \geq 0$ be the optimal threshold for the problem with free adjustment opportunities at rate $\lambda \geq 0$ but without fat-tailed shocks. Then consider the problem with fat-tailed shocks ξ that occur with Poisson rate $\lambda \geq 0$ but without free adjustment opportunities. Assume that the support of the large shocks satisfies $\underline{\xi} \geq \left(1+\sqrt{2}\right)\sqrt{\bar{y}}$. The optimal policy for this problem is a simple threshold policy with the same value $\bar{y}$ as in the problem with free adjustment opportunities.*

The previous proposition highlights the similarities in the determination of the threshold $\bar{y}$ between random menu cost and fat-tailed shocks. We remark that even if the policy is not simple, under mild conditions (such as independence across products) it will be of threshold type for the square norm of the vector; the difference is that characterization of the threshold requires a different analysis. The following results explore the difference implications for price changes for a given threshold.

Comparison of price changes. The distribution of price changes of a model with free-adjustment opportunities and one with fat-tailed shocks are different. The main difference is that in the fat tailed shock model considered above, every time that a ξ shock occur there are (some) *large* change in prices. Thus, fat-tailed shocks contribute to kurtosis mostly by having more frequent large shocks. Certainly, relative to the model with free-adjustment opportunities they contribute more to large price changes than to small price changes. In this context to have more frequent small price changes, the fat-tailed model relies on the multi product features, as in Midrigan (2011), as can be easily seen in the version with $n = 1$ in which the fat-tailed model has no small price changes. This implies that the peakeadness of the distribution of price changes around $\Delta p = 0$ is mostly determined by the multi-product feature of the model. For a more thorough analysis we need to add specify more about the distribution Γ.

Independent shock case. Assume that each of the component of ξ_i are independently drawn, and that satisfy equation (33). Thus, without loss of generality, assume that $\xi_i > 0$ with probability one for each product $i = 1, ..., n$. Note that in this case the lower bound of the support in each dimension satisfy:

$$\underline{\xi}_i \geq \left(1+\sqrt{2}\right)\sqrt{\bar{y}/n} \tag{34}$$

Note that in this case it is possible to have, in some component, very small price changes when there is a fat tail shock. Of course, if $\underline{\xi}_i$ is large enough that is not possible. In this case the marginal distribution is simpler to compute because it is the sum between the marginal distribution in the model with free adjustment opportunities (conditional on $||p||^2 < \bar{y}$ plus the (marginal) distribution of ξ_i.)

Proposition 10 *Assume that the fat-tailed shocks are independent across products and that $n \geq 3$. The resulting distribution of price changes is less peaked around small price changes when compared with the model with the same value of λ describing the arrival of free adjustment opportunities. In particular the level of the density is smaller around zero and the second derivative of this density around zero price changes is larger for the model with fat-tailed shocks.*

The proposition deals with the case of $n \geq 3$ because for $n = 2$ the distribution does not have a unique mode at $\Delta p_i = 0$, and indeed has density diverging to ∞ at values discreetly away from $\Delta p_i = 0$.

Lack of identification. The version of the model we wrote has more parameters than the equivalent model with free adjustment opportunities. In particular, there is a whole new function Γ. Because of this, without looking at more evidence it allows many more possibilities. To illustrate this we take it to a extreme and show a lack of identification result.

Proposition 11 *Let w be an arbitrary distribution of price changes. There are parameters ψ and a function Γ for which the model produces a distribution of price changes arbitrary close to the w.*

The proof of this proposition is trivial. Since we have shown that $\bar{y}$ is decreasing in the cost ψ and that $\bar{y}$ tends to zero as $\psi \downarrow 0$, then we let $\underline{\xi}$ be arbitrarily close to zero and allow almost all the price changes to happen a the time of large shocks, i.e. we can set $\Delta p = -\xi$. In other words, in a world with (almost) no menu cost, price changes will occur only because there are cost changes, and they will mirror them. Note that his is consistent with few price changes, because if cost changes happen infrequently so will price changes. This is clearly an extreme result, but highlights the need to think about identification of the objects on this version of the model.

Proofs.

Proof. (of Lemma 3)

$$\begin{aligned} ||p + \xi||^2 = \sum_{i=1}^{n} (p_i + \xi_i)^2 &= ||p||^2 + ||\xi||^2 + 2 \sum_{i=1}^{n} \xi_i \; p_i \\ &\geq ||p||^2 + ||\xi||^2 - 2\, ||p|| \, ||\xi|| \end{aligned}$$

where the inequality follows from the Cauchy-Schwarz inequality: $|\sum_{i=1}^{n} p_i \, \xi_i| \leq ||p|| \, ||\xi||$. Thus if $||p||^2 \leq \bar{y}$ and $||\xi|| \geq \kappa \sqrt{\bar{y}}$ then

$$
\begin{aligned}
||p + \xi||^2 &\geq ||p||^2 + ||\xi||^2 - 2 \, ||p|| \, ||\xi|| \geq ||\xi|| \, (||\xi|| - 2 \, ||p||) \\
&\geq \kappa \sqrt{\bar{y}} \left(\kappa \sqrt{\bar{y}} - 2 \sqrt{\bar{y}} \right) = \bar{y} \, \kappa \, (\kappa \, - 2)
\end{aligned}
$$

Hence taking $\kappa \geq 1 + \sqrt{2}$ we obtain the desired result. □

Proof. (of Proposition 9)We have shown that the solution of the value function for the problem with free adjustment opportunities but without the large shocks can be obtain by solving the value function $\hat{v}$ and the simple policy given by threshold y so that:

$$
(r + \lambda) \, \hat{v}(p) = B \, ||p||^2 + \frac{\sigma^2}{2} \sum_{i=1}^{n} \hat{v}_{ii}(p) + \lambda \hat{v}(0) \text{ for all } p : ||p||^2 \leq \bar{y} \tag{35}
$$

$$
\hat{v}(p) = v(0) + \psi \ , \quad \text{and } \hat{v}_i(p) = 0 \text{ for all } p : ||p||^2 = \bar{y} \tag{36}
$$

Now consider the problem without free adjustment opportunities but with fat-tailed shocks. We use a guess and verify strategy. The first part obtains a value function which satisfies the pde in the inaction and the boundary conditions using the same threshold $\bar{y}$. Here we use Lemma 3 which implies that every fat-tailed shocks takes the state out the inaction region and hence leads to an adjustment. Then the value function in inaction and boundary conditions are:

$$
(r + \lambda) \, v(p) = B \, ||p||^2 + \frac{\sigma^2}{2} \sum_{i=1}^{n} v_{ii}(p) + \lambda \, (v(0) + \psi) \text{ for all } p : ||p||^2 \leq \bar{y} \tag{37}
$$

$$
v(p) = v(0) + \psi \ , \quad \text{and } v_i(p) = 0 \text{ for all } p : ||p||^2 = \bar{y} \tag{38}
$$

That v solves equation (37) and equation (38) follows by setting $v(p) = \hat{v}(p) + a$. The only difference is that when the Poisson shock occurs the adjustment cost ψ is paid. Subtracting one equation from the other in the inaction region:

$$
(r + \lambda) a = \lambda \psi
$$

so that $a = -\lambda\psi/(r + \lambda)$ or $v(p) = \hat{v}(p) + \lambda\psi/(r + \lambda)$. Furthermore, the boundary conditions are also satisfied since the constant either does not affect the derivative or cancel in both sides of the equation. Finally, one can use the shape of the function $\hat{v}$, which is increasing in $||p||^2$, to show that the variational inequalities (31)-(32) are satisfied. □

Proof. (of Proposition 10) The proof proceed by obtain an expression for the marginal distribution of price changes for the case of fat-tailed shocks, and then examining both its second derivative and its level around zero price changes. First, consider the price changes conditional on $||p||^2 = y < \bar{y}$. This price changes have marginal distribution $\tilde{w}(x; y)$. To described this distribution we first introduce the distribution of the price gaps conditional on the norm square just before the large shock. As shown in the body of the paper it is given by

$$\omega(x_i; y) \;\; = \;\; \frac{1}{Beta\left(\frac{n-1}{2}, \frac{1}{2}\right)\; \sqrt{y}} \; \left(1 - \left(\frac{x_i}{\sqrt{\bar{y}}}\right)^2\right)^{(n-3)/2} \tag{39}$$

In the case of fat-tailed shocks the price changes is the sum of the price gap before the shock and the shock, namely $x_i + \xi_i$ and hence its distribution is given by:

$$\tilde{w}(\Delta p_i; y) = \int_{-\infty}^{\infty} \omega(\Delta p - x; y)\, \gamma(x)\, dx$$

where $\gamma = \Gamma_i'$ is the density of each of the coordinates of ξ_i. The second derivative of this conditional density evaluated at zero is:

$$\begin{aligned} \tilde{w}''(0; y) &= \int_{-\infty}^{\infty} \omega''(-x; y)\, \gamma(x)\, dx = \int_{-\min\{\underline{\xi}_i, \sqrt{y}\}}^{\min\{\underline{\xi}_i, \sqrt{y}\}} \omega''(-x; y)\, \gamma(x)\, dx \\ &= 2 \int_0^{\min\{\underline{\xi}_i, \sqrt{y}\}} \omega''(x; y)\, \gamma(x)\, dx > \omega''(0; y) \end{aligned} \tag{40}$$

where the second equation use the symmetry of $\tilde{w}$ and of γ around zero, as well as support of p_i and ξ_i. Note that for $n = 3$, the density $\tilde{w}$ is uniform, so its second derivative is zero everywhere. For $n \geq 3$ it has a peak at $\Delta p_i = 0$ with a strictly negative second derivative. For $n \geq 4$ the distribution is concave and then convex. The last inequality uses the properties described from $\tilde{w}$. The density of the distribution of price changes is given by

$$w(\Delta p_i) = \omega(\Delta p_i; \bar{y})\, (1 - \ell) + \left[\int_0^{\bar{y}} \tilde{w}(\Delta p_i; y) f(y) dy\right] \ell \qquad \text{for } n \geq 2 \;\; . \tag{41}$$

where we use the density of the price gaps f is independent of the fat-tailed shocks to price gaps and where ℓ has the same definition as in the body of the paper. Thus

$$\begin{aligned} w''(0) &= \omega''(0; \bar{y})\, (1 - \ell) + \left[\int_0^{\bar{y}} \tilde{w}''(0; y) f(y) dy\right] \ell \\ &> \omega''(0; \bar{y})\, (1 - \ell) + \left[\int_0^{\bar{y}} \omega''(0; y) f(y) dy\right] \ell\,, \end{aligned} \tag{42}$$

and thus the second derivative is larger for the model with fat-tailed shocks.

Finally, the same steps imply that the level of the density at zero is smaller with fat-tailed shocks, i.e.:

$$\begin{aligned} w(0) &= \omega(0; \bar{y})\, (1 - \ell) + \left[\int_0^{\bar{y}} \tilde{w}(0; y) f(y) dy\right] \ell \\ &< \omega(0; \bar{y})\, (1 - \ell) + \left[\int_0^{\bar{y}} \omega(0; y) f(y) dy\right] \ell\,, \end{aligned} \tag{43}$$

since

$$\tilde{w}(0;y) = \int_{-\infty}^{\infty} \omega(-x;y)\,\gamma(x)\,dx = \int_{-\min\{\underline{\xi}_i,\sqrt{y}\}}^{\min\{\underline{\xi}_i,\sqrt{y}\}} \omega(-x;y)\,\gamma(x)\,dx$$
$$= 2\int_{0}^{\min\{\underline{\xi}_i,\sqrt{y}\}} \omega(x;y)\,\gamma(x)\,dx < \omega(0;y) \tag{44}$$

where we use that $\omega(\cdot;y)$ is single peaked for $n \geq 3$. $\square$

G Proof that $\lim_{\bar{y}\to\infty} \xi(\sigma^2, r+\lambda, n, \bar{y}) = 0$

Note that, by examining the definition of κ_i and the sums in the expression for ξ we have that:

$$\lim_{\bar{y}\to\infty} \xi(\sigma^2, r+\lambda, n, \bar{y}) = \lim_{\bar{y}\to\infty} \xi\left(1, 1, n, \frac{(r+\lambda)\,\bar{y}}{\sigma^2}\right)$$

so this limit cannot depend on $r+\lambda$ or σ^2. Thus we denote it as:

$$\bar{\xi}(n) \equiv \lim_{\bar{y}\to\infty} \xi(1,1,n,\bar{y})$$

So we have:

$$\bar{y} \approx \frac{\psi}{B}\,(r+\lambda)\,\left[1-\bar{\xi}(n)\right] \quad \text{for large } \psi\,.$$

Now we show that $\bar{\xi}(n) = 0$. First we notice that the power series:

$$g(x) = \sum_{i=1}^{\infty}\prod_{s=1}^{i} \frac{1}{(s+2)(n+2s+2)}\; x^i$$

converges for all values of x since its coefficients satisfy the Cauchy-Hadamard inequality. Then we can write:

$$\xi(1,1,n,\bar{y}) \equiv \frac{\frac{2(n+2)}{\bar{y}}\frac{1}{g(\bar{y})} + \frac{1}{g(\bar{y})} + \frac{1}{\bar{y}^2}}{\frac{2(n+2)}{\bar{y}}\frac{1}{g(\bar{y})} + 2\frac{1}{g(\bar{y})} + \sum_{i=1}^{\infty}\omega(i,\bar{y})\;(2+i)}$$

where the weights $\omega(i,\bar{y})$ are given by:

$$\omega(i,x) = \frac{\frac{x^i}{\prod_{s=1}^{i}(s+2)(n+2s+2)}}{\sum_{j=1}^{\infty}\prod_{s=1}^{j}\frac{1}{(s+2)(n+2s+2)}\; x^j}$$

Note that for higher x the weights of smaller i decrease relative to the ones for higher i. Now since $g(\bar{y}) \to \infty$ as $\bar{y} \to \infty$, then:

$$\bar{\xi}(n) = \frac{1}{\lim_{\bar{y}\to\infty}\sum_{i=1}^{\infty}\omega(i,\bar{y})\;(2+i)}$$

To show that $\bar{\xi}(n) = 0$, suppose, by contradiction that is finite. Say, without loss of generality that equals $j+2$ for some integer j. Note that, by the form of the $\omega' s$ and because $g(\bar{y})$ diverges as $\bar{y}$ gets large enough, then by any j and $\epsilon > 0$ there exist a y^* large enough so that $\sum_{i=1}^{j} \omega(i, \bar{y}) < \epsilon$ for any $\bar{y} > y^*$. Thus, the expected value must be larger than $2 + j$.

Finally, we consider the case of $n \to \infty$. In this case we have that, the value function divided by n gives:

$$v = \min_T B \int_0^T \sigma^2 \, t \, e^{-(\lambda+r)} \, dt + e^{-(r+\lambda)T}(\Psi + v)$$

where $\Psi = \lim_{n\to\infty} \psi/n$. The first order condition for T gives, for a finite T:

$$0 = \big(B \, \sigma^2 \, T - (r+\lambda)\Psi\big) - (r+\lambda)e^{-(r+\lambda)T} v \tag{45}$$

Now consider the case where $\Psi \to \infty$. Note that v is finite since $T = \infty$, a feasible strategy as a finite value. Also let $\bar{Y} = \sigma^2 T = \lim_{n\to\infty} \frac{\bar{y}(n)}{n}$. Note that as $\Psi \to \infty$ then $\bar{Y}$ must also diverge towards ∞. Dividing the previous expression by Ψ:

$$\frac{\bar{Y}}{\Psi} = \frac{(r+\lambda)}{B} + (r+\lambda)e^{-(r+\lambda)T}\frac{v}{\Psi}$$

and taking the limits:

$$\lim_{\Psi\to\infty} \frac{\bar{Y}}{\Psi} = \frac{r+\lambda}{B} \ .$$

□

H Computing $\mathcal{M}$ and $kurt\,(\Delta p_i)$ for $n = 1$ and $n = \infty$

This section derives the closed form solution for the cumulated output effect and the kurtosis of price changes, $\mathcal{M}$ and $kurt\,(\Delta p_i)$, in two analytically tractable cases.

H.1 Analytical computation of $\mathcal{M}$ in the case of $n = 1$

We give an analytical summary expression for the effect of monetary shocks in two interesting cases, those for one product, i.e. $n = 1$, and those for the large number of product, i.e. $n = \infty$. The summary expression is the area under the impulse response for output, i.e. the sum of the output above steady state after a monetary shock of size $\delta > 0$, which we denote as:

$$\mathcal{M}_n(\delta) = (1/\epsilon) \int_0^\infty [\delta - \mathcal{P}_n(\delta, t)] \; dt \tag{46}$$

where ϵ is a the reciprocal of intertemporal elasticity of substitution, and where $\mathcal{P}_n(\delta, t)$ is the cumulative effect of monetary shock δ in the (log) of the price level after t periods. For large enough shocks, given the fixed cost of changing prices, the model display more price flexibility. Because of their preminence in the literature, and because of realism, we consider the case of small shocks δ by taking the first order approximation to equation (46), so we consider $\mathcal{M}_n(\delta) \approx \mathcal{M}_n'(0)\delta$.

For the case of $n = 1$ we obtain an analytical expression which, after normalizing by N_a depends only on λ/N_a. Thus as λ/N_a ranges from 0 to 1 the model ranges from a version of the menu cost model of *Golosov and Lucas* to a version using *Calvo* pricing. The analytical expression is based upon the following characterization:

$$\mathcal{M}_1(\delta) = (1/\epsilon) \int_{-\bar{p}}^{\bar{p}-\delta} m(p_0)\, h(p_0 + \delta)\, dp_0 \tag{47}$$

where p_0 is the price gap *after* the monetary shocks and where $m(p)$ gives the contribution to the area under the IRF of firms that start with price gap, after the shock, equal to p_0. Since the monetary shock happens when the economy is in steady state, the distribution right after the shock has the steady state density h displaced by δ. Immediately after the shock the firms with the highest price gap have price gap $\bar{p} - \delta$. Note that the integral in equation (47) does not include the firms that adjust on impact, those that before the shock have price gaps in the interval $[-\bar{p}, \bar{p} - \delta)$, whose adjustment does not contribute to the IRF. The definition of m is:

$$m(p) = -\mathbb{E}\left[\int_0^{\tau} p(t)\, dt \;\middle|\; p(0) = p\right] \tag{48}$$

where τ is the stopping time denoting the first time that the firm adjusts its price. This function gives the integral of the negative of the price gap until the first price adjustment. This expression is based on the fact that those firms with negative price gaps, i.e. low markups, contribute positively to output being in excess of its steady state value, and those with high markups contribute negatively. Given a decision rule summarized by $\bar{p}$ we can characterize m as the solution to the following ODE and boundary conditions:

$$\lambda m(p) = -p + \frac{\sigma^2}{2} m''(p) \;\text{ for all } p \in [-\bar{p}, \bar{p}] \;\text{ and }\; m(p) = 0 \;\text{ otherwise .} \tag{49}$$

The solution for the function m is:

$$m(p) = -\frac{p}{\lambda} + \frac{\bar{p}}{\lambda}\left(\frac{e^{\sqrt{2\phi}\frac{p}{\bar{p}}} - e^{-\sqrt{2\phi}\frac{p}{\bar{p}}}}{e^{\sqrt{2\phi}} - e^{-\sqrt{2\phi}}}\right) \;\text{ for all }\; p \in [-\bar{p}\,,\, \bar{p}] \;. \tag{50}$$

$\phi \equiv \lambda \bar{p}^2/\sigma^2$. We have then:

$$\mathcal{M}(\delta) \approx \mathcal{M}'(0)\delta = (\delta/\epsilon) \int_{\bar{p}}^{\bar{p}} m(p)\, h'(p)\, dp = (\delta/\epsilon)\, 2 \int_0^{\bar{p}} m(p)\, h'(p)\, dp \tag{51}$$

since $m(\bar{p})h(\bar{p}) = 0$. The last equality uses that m is negative symmetric, i.e. $m(p) = -m(-p)$, and that h is symmetric around zero. Using the expression for h in Section 3.1

$$h'(p) = -\frac{2\phi}{2\bar{p}^2\left(e^{\sqrt{2\phi}} - 1\right)^2}\left(e^{\sqrt{2\phi}\left(2 - \frac{p}{\bar{p}}\right)} + e^{\sqrt{2\phi}\frac{p}{\bar{p}}}\right) \quad \text{for} \quad p \in [0, \bar{p}] \;.$$

we obtain:

$$\begin{aligned}\mathcal{M}'(0)\delta &= \left(\frac{\delta}{\epsilon}\right)\frac{-2\phi}{\lambda\left(e^{\sqrt{2\phi}}-1\right)^2}\left(\frac{e^{\sqrt{2\phi}}\left(2+2\phi-2\cosh\left(\sqrt{2\phi}\right)\right)}{2\phi}\right)\\ &= \left(\frac{\delta}{\epsilon}\right)\frac{-2}{\lambda\left(e^{\sqrt{2\phi}}-1\right)^2}\left(e^{\sqrt{2\phi}}\left(1+\phi-\frac{e^{\sqrt{2\phi}}+e^{-\sqrt{2\phi}}}{2}\right)\right)\end{aligned}$$

Using the expression for N_a for the $n = 1$ and simple algebra we can rewrite it as:

$$\mathcal{M}'(0)\delta = \left(\frac{\delta}{\epsilon}\right)\frac{1}{N_a}\frac{e^{\sqrt{2\phi}}+e^{-\sqrt{2\phi}}}{\left(e^{\sqrt{2\phi}}+e^{-\sqrt{2\phi}}-2\right)^2}\left(e^{\sqrt{2\phi}}+e^{-\sqrt{2\phi}}-2-2\phi\right) \tag{52}$$

which yields the cumulated output effect of a small monetary shock of size δ.[49]

Kurtosis. We now verify that the expression can be equivalently obtained by computing the kurtosis, as stated in Proposition 7. For notation convenience let $x \equiv \sqrt{2\phi}$. Using the distribution of price changes derived in Section 3.1 and the definition of kurtosis we get

$$kurt\left(\Delta p_i\right) = \frac{2\ell\left(\frac{12}{x^4}-\frac{12+x^2}{x^2\left(e^{x/2}-e^{-x/2}\right)^2}\right)+1-\ell}{\left(2\ell\left(\frac{1}{x^2}+\frac{1}{2-e^{-x}-e^{x}}\right)+1-\ell\right)^2} = \frac{12-\frac{12x^2+x^4}{\left(e^{x/2}-e^{-x/2}\right)^2}+x^4\frac{1-\ell}{2\ell}}{2\ell\left(1+\frac{x^2}{2-e^{-x}-e^{x}}+x^2\frac{1-\ell}{2\ell}\right)^2}$$

Recall from Section 3.1 that $\ell = \frac{e^x+e^{-x}-2}{e^x+e^{-x}}$ so that , after some algebra

$$kurt\left(\Delta p_i\right) = 6\frac{e^x+e^{-x}}{\left(e^x+e^{-x}-2\right)^2}\left(e^x+e^{-x}-2-x^2\right)$$

It is immediate that the kurtosis and the cumulated effect on output shown in equation (52) satisfy Proposition 7.

H.2 Analytical computation of $\mathcal{M}$ in the case of $n = \infty$

Define

$$Y_n(t,\delta) \equiv \frac{1}{n}\sum_{i=1}^{n}[p_i(t)-\delta] = Y_n(t,0) - 2\delta\frac{\sum_{i=1}^n p_i(t)}{n} + \delta^2 \ .$$

[49] As a check of this formula compute the case for $\phi = 0$, i.e. the cumulated output for the Golosov-Lucas model. In this case we let $\lambda = 0$ and $\bar{p} > 0$. In this case we have: $m(p) = -\frac{\bar{p}^2 p}{3\sigma^2} + \frac{p^3}{3\sigma^2}$. Also $h'(p) = -1/\bar{p}^2$ for $p \in (0, \bar{p}]$, so we have:

$$\mathcal{M}'(0)\delta = \left(\frac{\delta}{\epsilon}\right)\frac{2}{-3\sigma^2\bar{p}^2}\int_0^{\bar{p}}\left[-\bar{p}^2 p + p^3\right]dp = \left(\frac{\delta}{\epsilon}\right)\frac{-2}{3\sigma^2\bar{p}^2}\left[-\frac{\bar{p}^4}{2}+\frac{\bar{p}^4}{4}\right] = \left(\frac{\delta}{\epsilon}\right)\frac{2\bar{p}^2}{3\sigma^2}\frac{2}{8} = \left(\frac{\delta}{\epsilon}\right)\frac{1}{N_a}\frac{1}{6}$$

which is the same value obtained by taking the limit for $\phi \to 0$ in the general expression above.

where the $p_i(t)$ are independent of each other, start at $p_i(0) = 0$ and have normal distribution with $\mathbb{E}[p_i(t)] = 0$ and $Var[p_i(t)] = \sigma^2 t$. Then, by an application of the law of large numbers, we have:

$$Y_\infty(t, \delta) = Y_\infty(t, 0) + \delta^2 = t\sigma^2 + \delta^2$$

Letting $\bar{Y} \equiv \lim_{n\to\infty} \bar{y}(n)/n$ we can represent the steady state optimal decision rule as adjusting prices when t, the time elapsed since last adjustment, attains $T = \bar{Y}/\sigma^2$. We compute the density of the distribution of products indexed by the time elapsed since the last adjustment t and, abusing notation, we denote it by f. This distribution is a truncated exponential with decay rate λ and with truncation T, thus the density is:

$$f(t) = \lambda \frac{e^{-\lambda t}}{1 - e^{-\lambda T}} \quad \text{for all} \quad t \in [0, T] \ .$$

The (expected) number of price changes per unit of time is given by the sum of the free adjustments and the ones that reach T, so

$$N_a = \lambda + f(T) = \lambda \left[1 + \frac{e^{-\lambda T}}{1 - e^{-\lambda T}}\right] = \frac{\lambda}{1 - e^{-\lambda T}}$$

Note that, using the definition of T given above, $\lambda T = \bar{Y}\lambda/\sigma^2$ the parameter which indexes the shape of f and of the distribution of price changes. Since this figures prominently in this expressions we define:

$$\phi \equiv \lambda T = \frac{\bar{Y}\,\lambda}{\sigma^2} \ .$$

which is consistent with the definition of ϕ in Proposition 3. Using this definition we get:

$$\ell = \frac{\lambda}{N_a} = 1 - e^{-\phi} \quad \text{and thus} \quad N_a = \frac{\lambda}{1 - e^{-\phi}}$$

Impulse Response of Prices to a monetary Shock. We can now define the impulse response. Note that after the monetary shock firms that have adjusted their prices t periods ago, in average will adjust their price up by δ. This highlights that as $n \to \infty$ there is no selection.

Now we turn to the characterization of the impact effect Θ. In this case we have

$$Y_\infty(t, \delta) = Y_\infty(t, 0) + \delta^2 = t\sigma^2 + \delta^2 \geq \bar{Y} = \sigma^2 T \iff t \geq T - \delta^2/\sigma^2 \ .$$

Thus the impact effect is:

$$\Theta(\delta) = \delta \int_{T-\delta^2/\sigma^2}^{T} f(t)dt = \delta \, \frac{e^{-\lambda T + \frac{\lambda}{\sigma^2}\delta^2} - e^{-\lambda T}}{1 - e^{-\kappa}} = \delta \, \frac{e^{-\kappa + \frac{\lambda}{\sigma^2}\delta^2} - e^{-\kappa}}{1 - e^{-\kappa}}$$

Using that $N_a Var[\Delta p_i] = \sigma^2$ we can write:

$$\Theta(\delta) = \delta + \delta\,\frac{e^{-\kappa + \frac{\lambda}{N_a}\frac{\delta^2}{Var[\Delta p_i]}} - 1}{1 - e^{-\kappa}} = \delta + \delta\,\frac{\left(1 - \frac{\lambda}{N_a}\right) e^{\frac{\lambda}{N_a}\frac{\delta^2}{Var[\Delta p_i]}} - 1}{\lambda/N_a}$$

Note that

$$\lim \Theta(\delta) = \begin{cases} \delta \left(\frac{\delta}{Std[\Delta p_i]}\right)^2 & \text{as } \lambda/N_a \to 0 \\ 0 & \text{as } \lambda/N_a \to 1 \end{cases}$$

and in general

$$\frac{\Theta(\delta)}{\partial(\lambda/N_a)} = \delta\,\frac{e^{\frac{\lambda}{N_a}\frac{\delta^2}{Var[\Delta p_i]}}\left(\frac{\delta^2}{Var[\Delta p_i]}\frac{\lambda}{N_a}\left(1 - \frac{\lambda}{N_a}\right) - 1\right) + 1}{(\lambda/N_a)^2} < 0$$

whenever $\delta < 2\,Std[\Delta p_i]$.

$$\begin{aligned}
\theta(t) &= \delta e^{-\lambda t}\left[f\left(T - \delta^2/\sigma^2 - t\right) + \lambda \int_0^{T-\delta^2/\sigma^2 - t} f(s)ds\right] \\
&= \delta e^{-\lambda t}\left[\lambda \frac{e^{-\lambda\left(T-\delta^2/\sigma^2-t\right)}}{1-e^{-\lambda T}} + \lambda \int_0^{T-\delta^2/\sigma^2-t} \lambda \frac{e^{-\lambda s}}{1-e^{-\lambda T}} ds\right] \\
&= \delta e^{-\lambda t}\left[\lambda \frac{e^{-\lambda\left(T-\delta^2/\sigma^2-t\right)}}{1-e^{-\lambda T}} + \lambda \frac{1 - e^{-\lambda(T-\delta^2/\sigma^2-t)}}{1-e^{-\lambda T}}\right] \\
&= \delta \frac{\lambda e^{-\lambda t}}{1-e^{-\lambda T}}\left[e^{-\lambda\left(T-\delta^2/\sigma^2-t\right)} + 1 - e^{-\lambda(T-\delta^2/\sigma^2-t)}\right] \\
&= \delta \frac{\lambda e^{-\lambda t}}{1-e^{-\lambda T}}
\end{aligned}$$

We can interpret $\theta(t)dt$ as $\theta(t)$ times the number of firms that adjust its price at times (t, dt). This is the sum of two terms. The first term is the fraction that adjust because they hit the boundary between t and $t + dt$. The second term is the fraction that have not yet adjusted times the fraction that adjust, λdt due to a free opportunity. Both terms are multiplied by $e^{-\lambda t}$ to take into account those firms that have received a free adjustment opportunity before after the monetary shock but before t.

Thus we have:

$$\begin{aligned}
\mathcal{P}_\infty(t, \delta) &= \Theta(\delta) + \delta \int_0^t \frac{\lambda e^{-\lambda s}}{1 - e^{-\lambda T}} ds = \Theta(\delta) + \delta \frac{1 - e^{-\lambda t}}{1 - e^{-\lambda T}} = \Theta(\delta) + \delta \frac{1 - e^{-\frac{\lambda}{N_a} t N_a}}{1 - e^{-\kappa}} \\
&= \Theta(\delta) + \delta \frac{1 - e^{-\frac{\lambda}{N_a} t N_a}}{\lambda/N_a}
\end{aligned}$$

Using $\mathcal{P}_\infty$ we can compute the IRF for output, and a summary measure for it, namely the area below it:

$$
\begin{aligned}
\mathcal{M}_\infty(\delta) &= \frac{1}{\epsilon}\int_0^T \left[\delta - \mathcal{P}_\infty(\delta,t)\right] dt \approx \delta\frac{1}{\epsilon}\int_0^T \left[1-\frac{1-e^{-\lambda t}}{1-e^{-\lambda T}}\right] dt \\
&= \frac{\delta}{\epsilon}\left[T - \frac{T}{1-e^{-\lambda T}} + \frac{1}{\lambda}\right] = \frac{\delta}{\epsilon}\left[-T\frac{e^{-\lambda T}}{1-e^{-\lambda T}} + \frac{1}{\lambda}\right] \\
&= \frac{\delta}{\epsilon}\frac{1-e^{-\lambda T}}{\lambda}\frac{1}{1-e^{-\lambda T}}\left[-\lambda T\frac{e^{-\lambda T}}{1-e^{-\lambda T}} + 1\right] \\
&= \frac{\delta}{\epsilon\, N_a}\frac{1}{1-e^{-\phi}}\left[1-\phi\,\frac{e^{-\phi}}{1-e^{-\phi}}\right] = \frac{\delta}{\epsilon\, N_a}\left[\frac{1-(1+\phi)\,e^{-\phi}}{\left(1-e^{-\phi}\right)^2}\right]
\end{aligned}
$$

where the approximation uses the expression for small δ, i.e. its first order Taylor's expansion.

Kurtosis. For completeness we also include here an expression for the kurtosis of the distribution of price changes in the case of $n=\infty$. Price changes are distributed as:

$$
\mathbb{E}\left[(\Delta p)^2\right] = \sigma^2/N_a = \frac{\sigma^2}{\lambda}\frac{\lambda}{N_a} = \frac{T\sigma^2}{T\lambda}\frac{\lambda}{N_a} = T\sigma^2\,\frac{1}{T\lambda}\frac{\lambda}{N_a}
$$

$$
\begin{aligned}
\mathbb{E}\left[(\Delta p)^4\right] &= 3\frac{\lambda}{N_a}\int_0^T \frac{(\sigma^2 t)^2\lambda e^{-\lambda t}}{1-e^{-\lambda T}} dt + \left(1-\frac{\lambda}{N_a}\right) 3\left(\sigma^2 T\right)^2 \\
&= 3\sigma^4\left[\lambda\int_0^T t^2 e^{-\lambda t} dt + \left(1-\frac{\lambda}{N_a}\right) T^2\right] \\
&= 3\sigma^4 T^2\left[\frac{2-e^{-\lambda T}\left(\lambda T\left(\lambda T+2\right)+2\right)}{\left(T\lambda\right)^2} + \left(1-\frac{\lambda}{N_a}\right)\right]
\end{aligned}
$$

Kurtosis is then given by:

$$
\begin{aligned}
\frac{\mathbb{E}\left[(\Delta p)^4\right]}{\left(\mathbb{E}\left[(\Delta p)^2\right]\right)^2} &= 3\,\frac{\frac{2-e^{-\lambda T}(\lambda T(\lambda T+2))}{(T\lambda)^2} + \left(1-\frac{\lambda}{N_a}\right)}{\left(\frac{1}{T\lambda}\right)^2\left(\frac{\lambda}{N_a}\right)^2} = 3\,\frac{2-e^{-\lambda T}\left(\lambda T\left(\lambda T+2\right)+2\right) + \left(T\lambda\right)^2\left(1-\frac{\lambda}{N_a}\right)}{\left(\frac{\lambda}{N_a}\right)^2} \\
&= 3\,\frac{\left(2-e^{-\lambda T}2\lambda T - e^{-\lambda T}2\right)}{\left(\frac{\lambda}{N_a}\right)^2} = 6\,\frac{\left(1-e^{-\lambda T}\left(1+\lambda T\right)\right)}{\left(\frac{\lambda}{N_a}\right)^2} = 6\,\frac{1-e^{-\lambda T}\left(1+\lambda T\right)}{\left(1-e^{-\lambda T}\right)^2} \\
&= 6\,\frac{1-e^{-\phi}\left(1+\phi\right)}{\left(1-e^{-\phi}\right)^2}
\end{aligned}
$$

It is immediate to use the expression for kurtosis and the one above for $\mathcal{M}_\infty(\delta)$ to verify Proposition 7.

I More model statistics

This appendix reports more model statistics that are functions only of n and ℓ. First we provide a formula to quantify the fraction of price changes that are smaller than a threshold $\kappa\mathbb{E}|\Delta p_i|$, which will prove useful to compare with the empirical evidence discussed above:

$$\mathcal{F}_n(\kappa) = 2 \int_0^{\kappa\mathbb{E}|\Delta p_i|} w(x) \; \mathrm{d}x$$

where $w(x)$ is density of price changes in equation (41).

Table 8: Model statistic for the fraction of price changes smaller than $\frac{1}{4}\,\mathbb{E}|\Delta p_i|$

% of free adjustments:	number of products n					
ℓ	1	2	4	6	10	50
0 %	0.00	0.10	0.13	0.14	0.15	0.16
10%	0.04	0.12	0.15	0.15	0.16	0.16
20%	0.08	0.13	0.16	0.16	0.17	0.17
50%	0.17	0.18	0.19	0.19	0.19	0.19
70%	0.20	0.20	0.20	0.20	0.20	0.21
80%	0.21	0.21	0.21	0.21	0.21	0.21
90%	0.22	0.22	0.22	0.22	0.22	0.22
95%	0.22	0.22	0.22	0.22	0.22	0.22

Table 9: Model statistic for $\mathbb{E}|\Delta p_i|/Std(\Delta p_i)$

% of free adjustments:	number of products n					
ℓ	1	2	4	6	10	50
0 %	1.00	0.90	0.85	0.83	0.82	0.80
10%	0.98	0.87	0.84	0.82	0.81	0.80
20%	0.95	0.86	0.83	0.81	0.80	0.79
50%	0.87	0.81	0.79	0.78	0.77	0.76
70%	0.81	0.77	0.76	0.75	0.75	0.75
80%	0.78	0.75	0.74	0.74	0.74	0.73
90%	0.74	0.73	0.73	0.73	0.72	0.72
95%	0.73	0.72	0.72	0.72	0.72	0.71

Table 10: Statistics by type of goods and outlet category (un-standardized price changes)

Good type	**Outlet type**	**Freq**	**Avg** $\lvert\Delta p\rvert$	**Std** $\lvert\Delta p\rvert$	**Kurt** (Δp_i)	**Frac25**
Food	Hypermakets	27.56	8.89	11.88	10.25	30.79
Food	Supermarkets	18.84	9.84	13.48	10.57	30.36
Food	Traditional	7.52	7.84	8.48	11.68	15.63
Food	Services	7.14	9.45	9.41	7.52	12.06
Durable goods	Hypermakets	15.82	13.35	12.97	6.36	21.02
Durable goods	Supermarkets	19.11	14.96	12.97	5.52	16.38
Durable goods	Traditional	7.93	14.77	15.82	7.08	22.02
Durable goods	Services	8.02	23.45	20.95	3.36	20.14
Clothing	Hypermakets	8.09	45.13	27.42	1.89	17.41
Clothing	Supermarkets	9.55	43.23	25.42	2.20	10.79
Clothing	Traditional	12.68	41.85	24.23	2.24	7.31
Clothing	Services	10.86	41.20	21.76	1.87	12.53
Other manufactured goods	Hypermakets	15.69	9.40	12.92	11.25	32.71
Other manufactured goods	Supermarkets	12.14	11.87	14.79	7.94	33.99
Other manufactured goods	Traditional	8.22	11.51	16.40	8.16	34.59
Other manufactured goods	Services	11.25	6.59	10.55	12.91	32.85
Energy	Hypermakets	80.89	3.56	2.84	9.23	8.28
Energy	Supermarkets	76.43	3.56	2.81	8.50	8.60
Energy	Traditional	75.55	4.22	3.51	5.39	14.35
Energy	Services	71.93	3.35	2.56	4.69	8.99
Services	Hypermakets	5.13	13.84	14.32	7.71	22.64
Services	Supermarkets	9.99	9.70	10.99	10.33	26.22
Services	Traditional	6.34	7.74	10.13	19.97	19.54
Services	Services	6.41	7.65	10.20	18.30	20.86

J More sectoral empirical results

Source is INSEE, monthly price records from French CPI, data from 2003:4 to 2011:4. Coverage is around 65% of CPI weight since rents, and prices of fresh food and centrally collected items (e.g. electricity, train and airplane tickets) are not included in the dataset. Freq. denotes monthly frequency of price change in percent. Size of price change Δp are the first-difference in the logarithm of price per unit, expressed in percent. Avg is average, Std standard deviation, Frac25 the share of absolute price change that are inferior to 0.25 Avg[$|\Delta p|$], Kurt denotes kurtosis. Observations with imputed prices or quality change are discarded. Moments are computed aggregating all prices changes using CPI weights at the product level.