Technical Appendix to Accompany Technological Revolutions and Stock Prices

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Technical Appendix

This appendix contains more detailed proofs than the ones sketched in the article.

Lemma 1: For later reference, we prove a more general version of Lemma 1. In particular, we cover three cases: (i) the new economy does not exist, and learning only occurs only by observing the old economy; (ii) the new economy exists, and learning occurs for $t \in [t^*, t^{**}]$; (iii) the new economy exists, adoption takes place at t^{**} and learning occurs for $t \geq t^{**}$. The learning dynamics for $t > t^{**}$ in the case of no adoption at t^{**} is identical to case (ii). For $t \geq t^*$ we then have

$$d\widehat{\psi}_t = \widehat{\sigma}_t^2 c \frac{\phi}{\sigma} d\widetilde{Z}_{0,t} + c_N \widehat{\sigma}_t^2 \frac{\phi}{\sigma_{N,1}} \left(1 - c\frac{\sigma_{N,0}}{\sigma}\right) d\widetilde{Z}_{1,t} \tag{B1}$$

$$\frac{d\widehat{\sigma}_t^2}{dt} = -\left(\widehat{\sigma}_t^2\right)^2 g \tag{B2}$$

where g, c and c_N are constants given by

$$g = \left(\left(\frac{c\phi}{\sigma}\right)^2 + c_N \left(\frac{\phi}{\sigma_{N,1}}\right)^2 \left(1 - c\frac{\sigma_{N,0}}{\sigma}\right)^2\right) \tag{B3}$$

$$(c, c_N) = \begin{cases} (1,0) & \text{if} \quad \text{only old economy exists} \\ (1,1) & \text{if} \quad t \geq t^{**} \text{ and adoption occurs at } t^{**} \\ (0,1) & \quad \text{otherwise} \end{cases} \tag{B4}$$

This implies that

$$\widehat{\sigma}_t^2 = \begin{cases} \left(\widehat{\sigma}_{t^{**}}^{-2} + g\left(t - t^{**}\right)\right)^{-1} & \text{if} \quad t \geq t^{**} \text{ and switch occurs at } t^{**} \\ \left(\widehat{\sigma}_{t^{*}}^{-2} + g\left(t - t^{*}\right)\right)^{-1} & \quad \text{otherwise} \end{cases} \tag{B5}$$

Proof: We consider only case (ii) and (iii). The simpler case (i) can be shown using similar steps. In these two cases, the new economy exists and thus the observation equations are

$$\begin{aligned} d\rho_t &= \phi\left(\overline{\rho} + c\psi - \rho_t\right) dt + \sigma_0 dZ_{0,t} \\ d\rho_t^N &= \phi\left(\overline{\rho} + \psi - \rho_t^N\right) dt + \sigma_{N,0} dZ_{0,t} + \sigma_{N,1} dZ_{1,t} \end{aligned}$$

where c is given in (B4). Defining $\mathbf{s}_t = (\rho_t, \rho_t^N)'$, this can be written compactly as

$$d\mathbf{s}_t = \left(\mathbf{A} + \mathbf{B}\mathbf{z} + \mathbf{C}\psi\right) dt + \mathbf{\Sigma} d\mathbf{Z}$$

where $\mathbf{C} = (c\phi, \phi)'$ and

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma & 0 \\ \sigma_{N,0} & \sigma_{N,1} \end{pmatrix}$$

Liptser and Shiryaev (1977) show that the process for $\widehat{\psi}_t = E_t[\psi]$ is given by

$$d\widehat{\psi}_t = \widehat{\sigma}_t^2 \mathbf{C}' \left(\mathbf{\Sigma}'\right)^{-1} d\widetilde{\mathbf{Z}} \tag{B6}$$

where $\widetilde{\mathbf{Z}}_t = (\widetilde{Z}_{0,t}, \widetilde{Z}_{1,t})'$ follows the process

$$d\widetilde{\mathbf{Z}}_t = \mathbf{\Sigma}^{-1} \left(\begin{array}{c} d\rho_t \\ d\rho_t^N \end{array} - E_t \left[\begin{array}{c} d\rho_t \\ d\rho_t^N \end{array} \right] \right)$$

and

$$\frac{d\widehat{\sigma}_t^2}{dt} = -\left(\widehat{\sigma}_t^2\right)^2 \mathbf{C}' \left(\mathbf{\Sigma}\mathbf{\Sigma}'\right)^{-1} \mathbf{C}$$

Substituting $\mathbf{C}$ and $\mathbf{\Sigma}$, we find immediately

$$\mathbf{C}' \left(\mathbf{\Sigma}'\right)^{-1} = \left(c\frac{\phi}{\sigma}, -c\phi\frac{\sigma_{N,0}}{\sigma\sigma_{N,1}} + \frac{\phi}{\sigma_{N,1}}\right)$$

Substituting this expression in (B6) and defining $g = \mathbf{C}' \left(\mathbf{\Sigma}\mathbf{\Sigma}'\right)^{-1} \mathbf{C}$ we obtain (B1) and (B2) for $c_N = 1$. It is simple to verify that (B5) satisfies (B2), yielding the conclusion. Q.E.D.

It is convenient to rewrite the original processes under the filtered measure. Let $b_t = \log(B_t)$ and $b_t^N = \log\left(B_t^N\right)$. For $t > t^*$ we have

$$db_t = \rho_t dt \quad \text{(B7)}$$

$$d\rho_t = \phi\left(\overline{\rho} + c\widehat{\psi}_t - \rho_t\right) dt + \sigma d\widetilde{Z}_{0,t} \quad \text{(B8)}$$

$$d\widehat{\psi} = \widehat{\sigma}_t^2 c\frac{\phi}{\sigma} d\widetilde{Z}_{0,t} + c_N \widehat{\sigma}_t^2 \frac{\phi}{\sigma_{N,1}}\left(1 - c\frac{\sigma_{N,0}}{\sigma}\right) d\widetilde{Z}_{1,t} \quad \text{(B9)}$$

$$d\widehat{\sigma}_t^2 = -\left(\widehat{\sigma}_t^2\right)^2 \left(\left(\frac{c\phi}{\sigma}\right)^2 + c_N\left(\frac{\phi}{\sigma_{N,1}}\right)^2\left(1 - c\frac{\sigma_{N,0}}{\sigma}\right)^2\right) dt \quad \text{(B10)}$$

$$db_t^N = \rho_t^N dt \quad \text{(B11)}$$

$$d\rho_t^N = \phi\left(\overline{\rho} + \widehat{\psi}_t - \rho_t^N\right) dt + \sigma_{N,0} d\widetilde{Z}_{0,t} + \sigma_{N,1} d\widetilde{Z}_{1,t} \quad \text{(B12)}$$

Lemma A1: Let $\tau = T - t$. The expectation in equation (6) is given by

$$V\left(B_t, \rho_t, \widehat{\psi}_t, \widehat{\sigma}_t^2, \tau\right) = E_t\left[\frac{B_T^{1-\gamma}}{1-\gamma}\right] = \frac{B_t^{1-\gamma}}{1-\gamma} e^{A_0(\tau) + (1-\gamma)A_1(\tau)\rho_t + (1-\gamma)A_2(\tau)\widehat{\psi}_t + \frac{1}{2}(1-\gamma)^2 A_2(\tau)^2 \widehat{\sigma}_t^2} \quad \text{(B13)}$$

where

$$A_0(\tau) = (1-\gamma)\overline{\rho}(\tau - A_1(\tau)) + \frac{\sigma^2}{2}\frac{(1-\gamma)^2}{\phi^2}\left\{\tau + \frac{1-e^{-2\phi\tau}}{2\phi} - 2\frac{1-e^{-\phi\tau}}{\phi}\right\}$$

$$A_1(\tau) = \frac{1-e^{-\phi\tau}}{\phi} \text{ and } A_2(\tau) = \tau - A_1(\tau)$$

Proof: By definition

$$V\left(b_t, \rho_t, \widehat{\psi}_t, \widehat{\sigma}_t^2, t; T\right) = (1-\gamma)^{-1} E_t\left[e^{(1-\gamma)b_T}\right]$$

Denoting $\mathbf{x}_t = \left(b_t, \rho_t, \widehat{\psi}_t, \widehat{\sigma}_t^2\right)$, the Feynman-Kac theorem shows that V has to satisfy the PDE

$$0 = \frac{\partial V}{\partial t} + \sum_i \frac{\partial V}{\partial x_i} E_t[dx_i] + \frac{1}{2}\sum_i\sum_j \frac{\partial^2 V}{\partial x_i \partial x_j} E_t[dx_i dx_j]$$

with boundary condition $V(\mathbf{x}_T) = (1-\gamma)^{-1} e^{(1-\gamma)x_{1,T}}$. Using (B7) - (B10) with $c = 1$ and $c_N = 0$, it is simple to verify that (B13) satisfies this PDE with the boundary condition. Finally, $A_2(\tau) > 0$

is immediate. Rewrite $A_2(\tau) = f(\tau) = \tau - \frac{1-e^{-\phi\tau}}{\phi}$. Note that $f(0) = 0$. Since $f'(\tau) = 1 - e^{-\phi\tau} > 0$, we have $f(\tau) > 0$ for every $\tau > 0$. QED.

Proof of Proposition 1: Since $\gamma > 1$ we have that V in (B13) is decreasing in $\widehat{\sigma}_t^2$. It immediately follows that $V\left(B_{t^*}(1-\kappa), \rho_{t^*}, 0, \widehat{\sigma}_{t^*}^2, \tau^*\right) < V\left(B_{t^*}, \rho_{t^*}, 0, 0, \tau^*\right)$. Q.E.D.

Proof of Proposition 2: Using (B13) it is immediate to verify that equation (13) follows from equation (14). Q.E.D.

To prove Proposition 3 we need the following lemmas, obtaining the closed form solution for the value function in equation (15) in the paper:

Lemma A2: The density of $\widehat{\psi}_{t^{**}}$ conditional on $\widehat{\psi}_t$ is normal and explicitly given by

$$\widehat{\psi}_{t^{**}}|_{\widehat{\psi}_t} \sim N\left(\widehat{\psi}_t, \sigma^2_{\widehat{\psi},t}\right)$$

where

$$\sigma^2_{\widehat{\psi},t} = \widehat{\sigma}_t^2 - \widehat{\sigma}_{t^{**}}^2$$

and $\widehat{\sigma}_t^2$ is given in (B5) for the case $t < t^{**}$.

*Proof***:** The process for the posterior mean $\widehat{\psi}_t$ is a linear diffusion with deterministic volatility, as given in (B1). The integral representation is

$$\widehat{\psi}_{t^{**}} = \widehat{\psi}_t + \frac{\phi}{\sigma_{N,1}} \int_t^{t^{**}} \widehat{\sigma}_s^2 d\widetilde{Z}_{1,s}$$

which immediately implies that

$$\widehat{\psi}_{t^{**}}|\widehat{\psi}_t \sim N\left(\widehat{\psi}_t, \sigma^2_{\widehat{\psi},t}\right)$$

where

$$\sigma^2_{\widehat{\psi},t} = \left(\frac{\phi}{\sigma_{N,1}}\right)^2 \int_t^{t^{**}} \left(\widehat{\sigma}_s^2\right)^2 ds$$

Using (B5) for $t < t^{**}$ we can compute

$$\int_t^{t^{**}} \left(\widehat{\sigma}_s^2\right)^2 ds = \frac{1}{\left(\phi/\sigma_{N,1}\right)^2}\left[\widehat{\sigma}_t^2 - \widehat{\sigma}_{t^{**}}^2\right]$$

Thus $\sigma^2_{\widehat{\psi},t} = \widehat{\sigma}_t^2 - \widehat{\sigma}_{t^{**}}^2$. In addition, it is then immediate that the probability of adoption is given by

$$p_t \equiv p\left(\widehat{\psi}_t, t\right) = \Pr\left(\widehat{\psi}_{t^{**}} > \underline{\psi}|\widehat{\psi}_t\right) = 1 - \mathcal{N}\left(\underline{\psi}; \widehat{\psi}_t, \sigma^2_{\widehat{\psi},t}\right)$$

where $\mathcal{N}\left(.; a, s^2\right)$ the cumulative density function of a normal distribution with mean a and variance s^2. Q.E.D.

Lemma A3: The value function

$$\mathcal{V}_t = E_t\left[\max_{yes,no} E_{t^{**}}\left[\frac{W_T^{1-\gamma}}{1-\gamma}\right]\right]$$

at time $t^* \leq t < t^{**}$ is given by

$$\mathcal{V}\left(B_t, \rho_t, \widehat{\psi}_t, \widehat{\sigma}_t^2; \tau\right) = \frac{B_t^{1-\gamma}}{1-\gamma}\left\{(1-p_t)\, G_t^{no} + p_t G_t^{yes}\right\} \tag{B14}$$

where

$$\begin{aligned} G_t^{no} &= e^{A_0(\tau)+(1-\gamma)A_1(\tau)\rho_t} \\ G_t^{yes} &= G_t^{no}\,(1-\kappa)^{1-\gamma} R_t e^{(1-\gamma)A_2(\tau^{**})\widehat{\psi}_t + \frac{1}{2}(1-\gamma)^2 A_2(\tau^{**})^2 \widehat{\sigma}_t^2} \end{aligned}$$

and

$$R_t = \frac{1-\mathcal{N}\left(\underline{\psi}; \widehat{\psi}_t + (1-\gamma)\, A_2(\tau^{**})\, \sigma^2_{\widehat{\psi},t}, \sigma^2_{\widehat{\psi},t}\right)}{1-\mathcal{N}\left(\underline{\psi}; \widehat{\psi}_t, \sigma^2_{\widehat{\psi},t}\right)} < 1 \tag{B15}$$

Proof: The value function is

$$\mathcal{V}_t = E_t\left[\max_{yes,no} E_{t^{**}}\left[\frac{W_T^{1-\gamma}}{1-\gamma}\right]\right] = (1-p_t)\, E_t\left[\frac{W_T^{1-\gamma}}{1-\gamma}|\widehat{\psi}_{t^{**}} < \underline{\psi}\right] + p_t E_t\left[\frac{W_T^{1-\gamma}}{1-\gamma}|\widehat{\psi}_{t^{**}} \geq \underline{\psi}\right]$$

as the adoption at t^{**} occurs if and only if $\widehat{\psi}_{t^{**}} \geq \underline{\psi}$. Starting with the first expectation, we can use the law of iterated expectations

$$E_t\left[\frac{W_T^{1-\gamma}}{1-\gamma}|\widehat{\psi}_{t^{**}} < \underline{\psi}\right] = E_t\left[E_{t^{**}}\left[\frac{W_T^{1-\gamma}}{1-\gamma}|\widehat{\psi}_{t^{**}} < \underline{\psi}\right]|\widehat{\psi}_{t^{**}} < \underline{\psi}\right]$$

We can use again equation (B13) to compute the inner expectation. In fact, if $\widehat{\psi}_{t^{**}} < \underline{\psi}$ the technology does not change at t^{**}. Moreover, eqn (B8) - (B9) show that ρ_t and $\widehat{\psi}_t$ are independent as $c = 0$ (see eqn. B4). Thus, Lemma A2 implies

$$E_{t^{**}}\left[\frac{W_T^{1-\gamma}}{1-\gamma}|\widehat{\psi}_{t^{**}} < \underline{\psi}\right] = V\left(B_{t^{**}}, \rho_{t^{**}}, 0, 0, t^{**}; T\right) = \frac{B_{t^{**}}^{1-\gamma}}{1-\gamma} e^{A_0(t^{**};T)+A_1(t^{**};T)\rho_{t^{**}}}$$

Thus,

$$\begin{aligned} E_t\left[\frac{B_{t^{**}}^{1-\gamma}}{1-\gamma} e^{A_0(t^{**};T)+A_1(t^{**};T)\rho_{t^{**}}}|\widehat{\psi}_{t^{**}} < \underline{\psi}\right] &= E_t\left[\frac{B_{t^{**}}^{1-\gamma}}{1-\gamma} e^{A_0(t^{**};T)+A_1(t^{**};T)\rho_{t^{**}}}\right] \\ &= \frac{B_t^{1-\gamma}}{1-\gamma} e^{A_0(t;T)+A_1(t;T)\rho_t} \end{aligned}$$

where the first equality stems from the independence of ρ_t and $\widehat{\psi}_t$, and the second equality stems from an application of Feynman - Kac thorem, similar to the argument used in Lemma A1.

The second expectation is more involved, as until t^{**} capital employs the old technology, and only then it switches to the new technology. In addition, the switch occurs only if $\widehat{\psi}_{t^{**}}$ is high enough, and this must be taken into account in the computation. Using again the law of iterated expectations, we have

$$\begin{aligned} E_t\left[\frac{W_T^{1-\gamma}}{1-\gamma}|\widehat{\psi}_{t^{**}} \geq \underline{\psi}\right] &= E_t\left[E_{t^{**}}\left[\frac{W_T^{1-\gamma}}{1-\gamma}|\widehat{\psi}_{t^{**}} > \underline{\psi}\right]|\widehat{\psi}_{t^{**}} > \underline{\psi}\right] \\ &= E_t\left[V\left(B_{t^{**}}(1-\kappa), \rho_{t^{**}}, \widehat{\psi}_{t^{**}}, \widehat{\sigma}^2_{t^{**}}, t^{**}; T\right)|\widehat{\psi}_{t^{**}} > \underline{\psi}\right] \end{aligned}$$

where the second equality stems from Lemma A1 and the fact that if $\widehat{\psi}_{t^{**}} > \underline{\psi}$, the adoption occurs. We can use the explicit formula for $V(.)$ to compute this expectation. In particular, from (B7) - (B10), $\widehat{\psi}_t$ is independent of both ρ_t and b_t, and $\widehat{\sigma}^2_{t^{**}}$ is a known constant. Thus, we can write

$$\begin{aligned} & E_t\left[V\left(B_{t^{**}}(1-\kappa), \rho_{t^{**}}, \widehat{\psi}_{t^{**}}, \widehat{\sigma}^2_{t^{**}}, t^{**}; T\right)|\widehat{\psi}_{t^{**}} > \underline{\psi}\right] \\ = & \frac{(1-\kappa)^{1-\gamma}}{1-\gamma}E_t\left[e^{(1-\gamma)b_{t**}+A_0(t^{**};T)+(1-\gamma)A_1(t^{**};T)\rho_{t**}+\frac{1}{2}(1-\gamma)^2A_2(t^{**};T)^2\widehat{\sigma}^2_{t^{**}}}\right] \\ & \times E_t\left[e^{(1-\gamma)A_2(t^{**};T)\widehat{\psi}_{t**}}|\widehat{\psi}_{t^{**}} > \underline{\psi}\right] \\ = & e^{(1-\gamma)b_t+A_0(t;T)+(1-\gamma)A_1(t;T)\rho_t+\frac{1}{2}(1-\gamma)^2A_2(t^{**};T)^2\widehat{\sigma}^2_{t^{**}}}E_t\left[e^{(1-\gamma)A_2(t^{**};T)\widehat{\psi}_{t**}}|\widehat{\psi}_{t^{**}} > \underline{\psi}\right] \end{aligned}$$

Since from Lemma A2, $\widehat{\psi}_{t^{**}} \sim N\left(\widehat{\psi}_t, \widehat{\sigma}^2_t - \widehat{\sigma}^2_{t^{**}}\right)$ we have that the conditional density required to compute the last expectation is given by

$$f\left(\widehat{\psi}_{t**}|\widehat{\psi}_{t**} > \underline{\psi}\right) = \frac{f\left(\widehat{\psi}_{t**}; \widehat{\psi}_t, \widehat{\sigma}^2_t - \widehat{\sigma}^2_{t^{**}}\right)1_{\{\widehat{\psi}_{t**}>\underline{\psi}\}}}{1-\mathcal{N}\left(\underline{\psi}; \widehat{\psi}_t, \widehat{\sigma}^2_t - \widehat{\sigma}^2_{t^{**}}\right)}$$

Using this density, we find

$$\begin{aligned} E\left[e^{(1-\gamma)A_2(t^{**};T)\widehat{\psi}_{t**}}|\widehat{\psi}_{t^{**}} > \underline{\psi}\right] &= \frac{1}{1-\mathcal{N}\left(\underline{\psi}; \widehat{\psi}_t, \widehat{\sigma}^2_t - \widehat{\sigma}^2_{t^{**}}\right)}\int_{\underline{\psi}}^{\infty} e^{(1-\gamma)A_2(t^{**};T)\widehat{\psi}_{t**}}f\left(\widehat{\psi}_{t**}\right)d\widehat{\psi}_{t^{**}} \\ &= e^{\frac{1}{2}(1-\gamma)^2A_2^2(\tau^{**})\left(\widehat{\sigma}^2_t - \widehat{\sigma}^2_{t^{**}}\right)+(1-\gamma)A_2(\tau^{**})\widehat{\psi}_t}R\left(\widehat{\psi}_t\right) \end{aligned}$$

where $R\left(\widehat{\psi}_t\right) = R_t$ is given in (B15). Putting all these elements together, we obtain (B14).

Lemma A4: $G_t^{yes} < G_t^{no}$.

Proof: Consider the expression

$$J_t = E_t\left[e^{(1-\gamma)\log(1-\kappa)+(1-\gamma)A_2(t^{**};T)\widehat{\psi}_{t**}+\frac{1}{2}(1-\gamma)^2A_2(t^{**};T)^2\widehat{\sigma}^2_{t^{**}}}|\widehat{\psi}_{t^{**}} > \underline{\psi}\right]$$

Using the definition of $\underline{\psi}$ in equation (13) of the paper, this can be written as

$$\begin{aligned} J_t &= E_t\left[e^{-(1-\gamma)A_2(\tau^{**})\left[-\frac{\log(1-\kappa)}{A_2(\tau^{*})}-\widehat{\psi}_{t**}-\frac{1}{2}(1-\gamma)A_2(t^{**};T)\widehat{\sigma}^2_{t^{**}}\right]}|\widehat{\psi}_{t^{**}} > \underline{\psi}\right] \\ &= E_t\left[e^{(1-\gamma)A_2(\tau^{**})\left[\widehat{\psi}_{t**}-\underline{\psi}\right]}|\widehat{\psi}_{t^{**}} > \underline{\psi}\right] \end{aligned}$$

Thus, $J_t < 1$, as it is the expectation of a random variable that is constrained to be less than 1. By using the same steps as in Lemma A3, we find

$$\begin{aligned} J_t &= E_t\left[e^{(1-\gamma)A_2(\tau^{**})\left[\widehat{\psi}_{t**}-\underline{\psi}\right]}|\widehat{\psi}_{t^{**}} > \underline{\psi}\right] \\ &= e^{-(1-\gamma)A_2(\tau^{**})\underline{\psi}}E_t\left[e^{(1-\gamma)A_2(\tau^{**})\widehat{\psi}_{t**}}|\widehat{\psi}_{t^{**}} > \underline{\psi}\right] \\ &= e^{-(1-\gamma)A_2(\tau^{**})\underline{\psi}+(1-\gamma)A_2(\tau^{*})\widehat{\psi}_t+\frac{1}{2}(1-\gamma)^2A_2(\tau^{*})^2\left(\widehat{\sigma}_t^2-\widehat{\sigma}_{t^{**}}^2\right)} \times R_t \\ &= e^{(1-\gamma)\log(1-\kappa)+(1-\gamma)A_2(\tau^{*})\widehat{\psi}_t+\frac{1}{2}(1-\gamma)^2A_2(\tau^{*})^2\widehat{\sigma}_t^2} \times R_t \\ &= \frac{G_t^{yes}}{G_t^{no}} \end{aligned}$$

yielding the conclusion. Q.E.D.

Proposition 3: Experimenting is always optimal at time t^*, that is

$$\mathcal{V}\left(B_{t^*}, \rho_{t^*}, 0, \widehat{\sigma}_{t^*}^2; \tau^*\right) > V\left(B_{t^*}, \rho_{t^*}, 0, 0; \tau^*\right)$$

where $V\left(B_{t^*}, \rho_{t^*}, 0, 0; \tau^*\right)$ is defined in equation (B13).

Proof: Since $G_t^{yes} < G_t^{no}$, the result follows from the fact that we can rewrite $V\left(B_{t^*}, \rho_{t^*}, 0, 0; \tau^*\right) = \frac{B_{t^*}^{1-\gamma}}{1-\gamma}G_{t^*}^{no}$ and $\gamma > 1$. Q.E.D.

Proof of Proposition 4: The proof is identical to the one of Lemma A3, where "$(1-\gamma)$" is substituted with "$-\gamma$". Using this fact, we have

$$\pi_t = \lambda^{-1}B_t^{-\gamma}\left\{(1-p_t)\,\widetilde{G}_t^{no} + p_t\widetilde{G}_t^{yes}\right\} \tag{B16}$$

where

$$\widetilde{G}_t^{no} = e^{\overline{A}_0(\tau)-\gamma A_1(\tau)\rho_t} \tag{B17}$$

$$\widetilde{G}_t^{yes} = \widetilde{G}_t^{no}\,(1-\kappa)^{-\gamma}\,\widetilde{R}_t e^{-\gamma A_2(\tau^{**})\widehat{\psi}_t+\frac{1}{2}\gamma^2A_2(\tau^{**})^2\widehat{\sigma}_t^2} \tag{B18}$$

and

$$\widetilde{R}_t = \frac{1-\mathcal{N}\left(\underline{\psi};\widehat{\psi}_t-\gamma A_2\left(\tau^{**}\right)\sigma_{\widehat{\psi},t}^2,\sigma_{\widehat{\psi},t}^2\right)}{1-\mathcal{N}\left(\underline{\psi};\widehat{\psi}_t,\sigma_{\widehat{\psi},t}^2\right)} < 1 \tag{B19}$$

In this proposition,

$$\overline{A}_0(\tau) = -\gamma\overline{\rho}\left(\tau - A_1(\tau)\right) + \frac{\sigma^2}{2}\frac{\gamma^2}{\phi^2}\left\{\tau + \frac{1-e^{-2\phi\tau}}{2\phi} - 2\frac{1-e^{-\phi\tau}}{\phi}\right\}$$

Q.E.D.

Proof of Corollary 1: The corollary follows from an application of Ito's Lemma, so that

$$\frac{d\pi_t}{\pi_t} = -\boldsymbol{\sigma}_{\pi,t}d\widetilde{\mathbf{Z}}_t$$

where

$$\boldsymbol{\sigma}_{\pi,t} = \gamma A_1(\tau)\,\boldsymbol{\sigma} + S_{\pi,t}\widetilde{\boldsymbol{\sigma}}_{\psi,t}$$

and

$$S_{\pi,t} = \frac{\left(\gamma A_2(\tau^{**}) - \frac{1}{\widetilde{p}}\frac{\partial\widetilde{p}}{\partial\widehat{\psi}}\right)\widetilde{G}_t^{yes} + \frac{\partial p}{\partial\widehat{\psi}}\widetilde{G}_t^{no}}{(1-p_t)\,\widetilde{G}_t^{no} + p_t\widetilde{G}_t^{yes}} \tag{B20}$$

where

$$\widetilde{p}_t \equiv \widetilde{p}\left(\widehat{\psi}_t, t\right) = 1 - \mathcal{N}\left(\underline{\psi}; \widehat{\psi}_t - \gamma A_2(\tau^{**})\,\sigma^2_{\widehat{\psi},t}, \sigma^2_{\widehat{\psi},t}\right)$$

and $\boldsymbol{\sigma} = (\sigma, 0)$, $\widetilde{\boldsymbol{\sigma}}_{\psi} = \left(0, \widehat{\sigma}_t^2 \frac{\phi}{\sigma_{N,1}}\right)$. Q.E.D.

Proof of Proposition 5 (old economy): The result about the old economy is immediate from the pricing formula $M_t = E_t[\pi_T B_T]/\pi_t = E_t\left[B_T^{1-\gamma}\right]/\pi_t$, and the results in Lemma A3 and Proposition 4. Q.E.D.

For better referencing, it is convenient to restate Proposition 5 for the new economy:

Proposition 5 (new economy) Let $\tau = T - t$. For $t^* \le t < t^{**}$, the market to book ratio of the new economy is given by

$$\frac{M_t^N}{B_t^N} = \frac{(1-p_t)\,K^{no} + p_t K^{yes}}{(1-p_t)\,\widetilde{G}_t^{no} + p_t\widetilde{G}_t^{yes}} \tag{B21}$$

where $\widetilde{G}_t^{no}$ and $\widetilde{G}_t^{yes}$ are given in Proposition 3, and

$$\begin{aligned} K^{no} &= K_t R_{L,t}^N \\ K^{yes} &= (1-\kappa)^{-\gamma} K_t^N R_{H,t}^N \end{aligned}$$

$$\begin{aligned} K_t &= e^{C_0(\tau) - \gamma A_1(\tau)\rho_t + A_1(\tau)\rho_t^N + A_2(\tau)\widehat{\psi}_t + \frac{1}{2}A_2^2(\tau)\widehat{\sigma}_t^2} \\ K_t^N &= K_t e^{-\gamma A_2(\tau^{**})\widehat{\psi}_t + \frac{1}{2}\gamma A_2(\tau^{**})(\gamma A_2(\tau^{**}) - 2A_2(\tau))\widehat{\sigma}_t^2} \end{aligned}$$

and

$$\begin{aligned} R_{L,t}^N &= \frac{\mathcal{N}\left(\underline{\psi}; \widehat{\psi}_t + \sigma^L_{y\widehat{\psi}}, \sigma^2_{\widehat{\psi},t}\right)}{\mathcal{N}\left(\underline{\psi}; \widehat{\psi}_t, \sigma^2_{\widehat{\psi},t}\right)} \text{ with } \sigma^L_{y\widehat{\psi}} = A_2(\tau)\,\widehat{\sigma}_t^2 - A_2(\tau^{**})\,\widehat{\sigma}_{t^{**}}^2 \\ R_{H,t}^N &= \frac{1 - \mathcal{N}\left(\underline{\psi}; \widehat{\psi}_t + \sigma^H_{y\widehat{\psi}}, \sigma^2_{\widehat{\psi},t}\right)}{1 - \mathcal{N}\left(\underline{\psi}; \widehat{\psi}_t, \sigma^2_{\widehat{\psi},t}\right)} \text{ with } \sigma^H_{y\psi} = \sigma^L_{y\psi} - \gamma A_2(\tau^{**})\,\sigma^2_{\widehat{\psi},t} \end{aligned}$$

Above, $C_0(\tau)$ is given by

$$\begin{aligned} C_0(\tau) &= (1-\gamma)\,\overline{\rho}\,(\tau - A_1(\tau)) \\ &\quad + \frac{1}{2\phi^2}\left\{\tau + \frac{1 - e^{-2\phi\tau}}{2\phi} - 2\frac{1 - e^{-\phi\tau}}{\phi}\right\}\left(\gamma^2\sigma^2 - 2\gamma\sigma_{N,0}\sigma + \left(\sigma^2_{N,0} + \sigma^2_{N,1}\right)\right) \end{aligned}$$

We start the proof with two lemmas:

Lemma A5: For $t \geq t^{**}$, let $\tau = T - t$. Then

$$V^N\left(b_t, b_t^N, \rho_t, \rho_t^N, \widehat{\psi}_t, \widehat{\sigma}_t^2, \tau\right) \equiv E_t\left[e^{-\gamma b_T + b_T^N}\right]$$

is given by

$$V^N\left(b_t, b_t^N, \rho_t, \rho_t^N, \widehat{\psi}_t, \widehat{\sigma}_t^2, \tau\right) = e^{-\gamma b_t + b_t^N + C_0(\tau) - \gamma A_1(\tau)\rho_t + A_1(\tau)\rho_t^N + A_2(\tau)\widehat{\psi}_t + \frac{1}{2}(1-c\gamma)^2 A_2^2(\tau)\widehat{\sigma}_t^2} \quad \text{(B22)}$$

where $c = 1$ if the adoption occurred at time t^{**} , and 0 otherwise, $A_1(.)$ and $A_2(.)$ are as in Lemma A1, and

$$C_0(\tau) = (1-\gamma)\,\overline{\rho}\,(\tau - A_1(\tau)) + \frac{1}{\phi^2}\left(\tau + \frac{1 - e^{-2\phi\tau}}{2\phi} - 2A_1(\tau)\right)\frac{1}{2}(\sigma^*)^2$$

and

$$(\sigma^*)^2 = \gamma^2\sigma^2 + \sigma_{N,0}^2 + \sigma_{N,1}^2 - 2\gamma\sigma_{N,0}\sigma$$

Proof: As in Lemma A1, denoting $\mathbf{x}_t = \left(b_t, b_t^N, \rho_t, \rho_t^N, \widehat{\psi}_t, \widehat{\sigma}_t^2\right)$, the Feynman-Kac theorem shows that V^N has to satisfy the PDE

$$0 = \frac{\partial V^N}{\partial t} + \sum_i \frac{\partial V^N}{\partial x_i} E_t[dx_i] + \frac{1}{2}\sum_i \sum_j \frac{\partial^2 V^N}{\partial x_i \partial x_j} E_t[dx_i dx_j]$$

with the boundary condition $V(\mathbf{x}_T) = (1-\gamma)^{-1} e^{-\gamma x_{1,T} + x_{2,T}}$. Using (B7) - (B12) for the cases where $c = 1$ or $c = 0$ (with $c_N = 1$) in Lemma 1, it is simple to verify that (B22) satisfies this PDE with the boundary condition provided. Q.E.D.

Lemma A6: Define

$$y_{t^{**}} = -\gamma b_{t^{**}} + b_{t^{**}}^N - \gamma A_1(\tau^{**})\,\rho_{t^{**}} + A_1(\tau^{**})\,\rho_{t^{**}}^N + (1 - c_1)\,A_2(\tau^{**})\,\widehat{\psi}_{t^{**}}$$

where $c_1 > 0$ is a constant. Then

$$\begin{pmatrix} y_{t^{**}} \\ \widehat{\psi}_{t^{**}} \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_{y,t} \\ \widehat{\psi}_t \end{pmatrix}, \begin{pmatrix} \sigma_y^2 & \sigma_{y\psi} \\ \sigma_{y\psi} & \sigma_{\widehat{\psi},t}^2 \end{pmatrix}\right)$$

where

$$\begin{aligned}
\mu_{y,t} &= -\gamma b_t + b_t^N + (1-\gamma)\,\overline{\rho} a(t) - \gamma A_1(\tau)\,\rho_t + A_1(\tau)\,\rho_t^N + (A_2(\tau) - c_1 A_2(\tau^{**}))\,\widehat{\psi}_t \\
\sigma_y^2 &= (1-c_1)^2 A_2(\tau^{**})^2\left(\widehat{\sigma}_t^2 - \widehat{\sigma}_{t^{**}}^2\right) + a(t)^2\,\widehat{\sigma}_t^2 + 2A_2(\tau^{**})\,(1-c_1)\,a(t)\,\widehat{\sigma}_t^2 + (\sigma^*)^2\,a_2(t) \\
\sigma_{\widehat{\psi},t}^2 &= \widehat{\sigma}_t^2 - \widehat{\sigma}_{t^{**}}^2 \\
\sigma_{y\psi} &= (1-c_1)\,A_2(\tau^{**})\left(\widehat{\sigma}_t^2 - \widehat{\sigma}_{t^{**}}^2\right) + a(t)\,\widehat{\sigma}_t^2
\end{aligned}$$

and

$$a(t) = t^{**} - t - \frac{e^{-\phi(T-t^{**})} - e^{-\phi(T-t)}}{\phi} \tag{B23}$$

$$a_2(t) = \frac{1}{\phi^2}\left(t^{**} - t + \frac{e^{-2\phi(T-t^{**})} - e^{-2\phi(T-t)}}{2\phi} - 2\frac{e^{-\phi(T-t^{**})} - e^{-\phi(T-t)}}{\phi}\right) \tag{B24}$$

$$(\sigma^*)^2 = \gamma^2\sigma^2 + \sigma_{N,0}^2 + \sigma_{N,1}^2 - 2\gamma\sigma_{N,0}\sigma \tag{B25}$$

Proof: The proof of this lemma is rather lengthy, and so it is provided separately below.

Proof of Proposition 5 (new economy): The pricing formula is $M_t^N = E_t\left[\pi_T B_T^N\right]/\pi_t$. Thus, we need to compute

$$E_t\left[B_T^{-\gamma} B_T^N\right] = (1 - p_t)\, E_t\left[B_T^{-\gamma} B_T^N | \widehat{\psi}_{t^{**}} < \underline{\psi}\right] + p_t E_t\left[B_T^{-\gamma} B_T^N | \widehat{\psi}_{t^{**}} > \underline{\psi}\right] \tag{B26}$$

Starting with the first expectation, note that if $\widehat{\psi}_{t^{**}} < \underline{\psi}$, no adoption occurs at t^{**}. Thus,

$$\begin{aligned}
E_t\left[B_T^{-\gamma} B_T^N | \widehat{\psi}_{t^{**}} < \underline{\psi}\right] &= E_t\left[E_{t^{**}}\left[B_T^{-\gamma} B_T^N | \widehat{\psi}_{t^{**}} < \underline{\psi}\right] | \widehat{\psi}_{t^{**}} < \underline{\psi}\right] \\
&= E_t\left[V^N\left(b_{t^{**}}, b_{t^{**}}^N, \rho_{t^{**}}, \rho_{t^{**}}^N, \widehat{\psi}_{t^{**}}, \widehat{\sigma}_{t^{**}}^2, t^{**}; T\right) | \widehat{\psi}_{t^{**}} < \underline{\psi}\right] \\
&= e^{C_0(\tau^{**}) + \frac{1}{2}A_2^2(\tau^{**})\widehat{\sigma}_{t^{**}}^2} \\
&\quad \times E_t\left[e^{-\gamma b_{t^{**}} + b_{t^{**}}^N - \gamma A_1(\tau^{**})\rho_{t^{**}} + A_1(\tau^{**})\rho_{t^{**}}^N + A_2(\tau^{**})\widehat{\psi}_{t^{**}}} | \widehat{\psi}_{t^{**}} < \underline{\psi}\right]
\end{aligned}$$

where the first equality stems from the law of iterated expectations, the second from the fact that $\widehat{\psi}_{t^{**}}$ is known at t^{**}, the third from Lemma A5, with $c = 0$ as the adoption does not occur at t^{**}. Note that the exponent in the expectation is simply $y_{t^{**}}$ in Lemma A6 with $c_1 = 0$. For notational convenience, let

$$a_0(t) = (1 - \gamma)\,\overline{\rho} a(t).$$

Using Lemma A6 with $c_1 = 0$ and denoting by L the corresponding quantities in Lemma A6 for this case, we can compute

$$E\left[e^{y_{t^{**}}} | \widehat{\psi}_{t^{**}} < \underline{\psi}\right] = \frac{\int_{-\infty}^{\underline{\psi}} E\left[e^{y_{t^{**}}} | \widehat{\psi}_{t^{**}}\right] f\left(\widehat{\psi}_{t^{**}}; \widehat{\psi}_t, \sigma_{\widehat{\psi},t}^2\right) d\widehat{\psi}_{t^{**}}}{\Pr\left(\widehat{\psi}_{t^{**}} < \underline{\psi}\right)}$$

where $f\left(\widehat{\psi}_{t^{**}}; \widehat{\psi}_t, \sigma_{\widehat{\psi},t}^2\right)$ is the density of a normal with mean $\widehat{\psi}_t$ and variance $\sigma_{\widehat{\psi},t}^2$. The rules of the conditional normal distribution yield the following expression for this expectation:

$$E_t\left[e^{y_{t^{**}}} | \widehat{\psi}_{t^{**}} < \underline{\psi}\right] = B_t^{-\gamma} B_t^N e^{a_0(t) - \gamma A_1(t;T)\rho_t + A_1(t;T)\rho_t^N + A_2(t;T)\widehat{\psi}_t + \frac{1}{2}\sigma_{Ly}^2} R_{L,t}^N$$

where $R_{L,t}^N$ is given in Proposition 5. So, finally, the first expectation is given by

$$\begin{aligned}
E_t\left[B_T^{-\gamma} B_T^N | \widehat{\psi}_{t^{**}} < \underline{\psi}\right] &= B_t^{-\gamma} B_t^N e^{C_0(t^{**};T) + \frac{1}{2}A_2^2(t^{**};T)\widehat{\sigma}_{t^{**}}^2} e^{a_0(t) - \gamma A_1(t;T)\rho_t + A_1(t;T)\rho_t^N + A_2(t;T)\widehat{\psi}_t + \frac{1}{2}\sigma_{L,y}^2} R_{L,t}^N \\
&= B_t^{-\gamma} B_t^N e^{C_0(t;T) + \frac{1}{2}A_2^2(t;T)\widehat{\sigma}_t^2 - \gamma A_1(t;T)\rho_t + A_1(t;T)\rho_t^N + A_2(t;T)\widehat{\psi}_t} R_{L,t}^N
\end{aligned}$$

where the second equality is obtained from the first after some tedious algebra.

We now turn to the second expectation in (B26). The methodology is the same as before, although now we must set $c=1$ in (B22) and note that $B_{t^{**}} = (1-\kappa) B_{t^{**}_{-}}$, which implies $b_{t^{**}} = b_{t^{**}_{-}} + \log(1-\kappa)$. Specifically, we have that for $t \leq t^{**}$

$$\begin{aligned}
E_t\left[B_T^{-\gamma} B_T^N | \widehat{\psi}_{t^{**}} \geq \underline{\psi}\right] &= E_t\left[V^N\left(\log(1-\kappa) + b_{t^{**}_{-}}, b^N_{t^{**}}, \rho_{t^{**}}, \rho^N_{t^{**}}, \widehat{\psi}_{t^{**}}, \widehat{\sigma}^2_{t^{**}}, \tau\right) | \widehat{\psi}_{t^{**}} \geq \underline{\psi}\right] \\
&= (1-\kappa)^{-\gamma} e^{C_0(\tau^{**}) + \frac{1}{2}(1-\gamma)^2 A_2^2(\tau^{**}) \widehat{\sigma}^2_{t^{**}}} \times \\
&\quad \times E_t\left[e^{-\gamma b_{t^{**}} + b^N_{t^{**}} - \gamma A_1(\tau^{**}) \rho_{t^{**}} + A_1(\tau^{**}) \rho^N_{t^{**}} + (1-\gamma) A_2(\tau^{**}) \widehat{\psi}_{t^{**}}} | \widehat{\psi}_{t^{**}} \geq \underline{\psi}\right]
\end{aligned}$$

Comparing to the case with $\left\{\widehat{\psi}_{t^{**}} < \underline{\psi}\right\}$, we see that the term in the expectation is identical, but for the coefficient of $\widehat{\psi}_{t^{**}}$, which is multiplied by $(1-\gamma)$. The distribution of the exponent is given in Lemma A6 for $c_1 = \gamma$. In this case, defining

$$y_{H,t^{**}} = -\gamma b_{t^{**}} + b^N_{t^{**}} - \gamma A_1(\tau^{**}) \rho_{t^{**}} + A_1(\tau^{**}) \rho^N_{t^{**}} + (1-\gamma) A_2(\tau^{**}) \widehat{\psi}_{t^{**}}$$

we have that

$$\mu_{H,y,t} = E\left[y_{H,t^{**}}\right] = -\gamma b_t + b_t^N + a_0(t) - \gamma A_1(\tau) \rho_t + A_1(\tau) \rho_t^N + (A_2(\tau) - \gamma A_2(\tau^{**})) \widehat{\psi}_t$$

The same steps then show

$$\begin{aligned}
E_t\left[e^{y_{H,t^{**}}} | \widehat{\psi}_{t^{**}} > \underline{\psi}\right] &= \frac{1}{1 - N\left(\underline{\psi}; \widehat{\psi}_t, \sigma^2_{\widehat{\psi}}\right)} \int_{\underline{\psi}}^{\infty} E\left[e^{y_{H,t^{**}}} | \widehat{\psi}_{t^{**}}\right] f\left(\widehat{\psi}_{t^{**}}\right) d\widehat{\psi}_{t^{**}} \\
&= B_t^{-\gamma} B_t^N e^{a_0(t) - \gamma A_1(t;T)\rho_t + A_1(t;T)\rho_t^N + (A_2(\tau;T) - \gamma A_2(t^{**};T))\widehat{\psi}_t + \frac{1}{2}\sigma^2_{Hy}} R^N_{H,t}
\end{aligned}$$

where $R^N_{H,t}$ is defined in Proposition 5.

So, we finally obtain

$$\begin{aligned}
E_t\left[B_T^{-\gamma} B_T^N | \widehat{\psi}_{t^{**}} \geq \underline{\psi}\right] &= B_t^{-\gamma} B_t^N (1-\kappa)^{-\gamma} e^{C_0(\tau^{**}) + \frac{1}{2}(1-\gamma)^2 A_2(\tau^{**})^2 \widehat{\sigma}^2_{t^{**}}} \\
&\quad \times e^{a_0(t) - \gamma A_1(\tau)\rho_t + A_1(\tau)\rho_t^N + (A_2(\tau) - \gamma A_2(\tau^{**}))\widehat{\psi}_t + \frac{1}{2}\sigma^2_{Hy}} R^N_{H,t} \\
&= B_t^{-\gamma} B_t^N (1-\kappa)^{-\gamma} e^{C_0(\tau) - \gamma A_1(\tau)\rho_t + A_1(\tau)\rho_t^N + (A_2(\tau) - \gamma A_2(\tau^{**}))\widehat{\psi}_t + \frac{1}{2}(A_2(\tau) - \gamma A_2(\tau^{**}))^2 \widehat{\sigma}_t^2} R^N_{H,t}
\end{aligned}$$

where the second equality is obtained from the first after some tedious algebra. Putting all terms together, we obtain the expression in Proposition 5. Q.E.D.

Proof of Corollary 2: The proof follows from an application of Ito's Lemma to the respective pricing functions. We obtain

$$\boldsymbol{\sigma}^N_M = A_1(\tau) \boldsymbol{\sigma}_N + \left(S^N_{M,t} + S_{\pi,t}\right) \widetilde{\boldsymbol{\sigma}}_\psi$$

where $\boldsymbol{\sigma}_N = (\sigma_{N,0}, \sigma_{N,1})$ and

$$S^N_{M,t} = \frac{\left(A_2(\tau) + \frac{1}{p^N_{L,t}} \frac{\partial p^N_{L,t}}{\partial \widehat{\psi}}\right) K^{no}_t + \left((A_2(\tau) - \gamma A_2(\tau^{**})) + \frac{1}{p^N_{H,t}} \frac{\partial p^N_{H,t}}{\partial \widehat{\psi}}\right) K^{yes}_t}{(1 - p_t) K^{no}_t + p_t K^{yes}_t} \tag{B27}$$

with

$$p^N_{L,t} \equiv p^N_L\left(\widehat{\psi}_t, t\right) = \mathcal{N}\left(\underline{\psi}; \widehat{\psi}_t + \sigma^L_{y\widehat{\psi}}, \sigma^2_{\widehat{\psi},t}\right) \tag{B28}$$

$$p^N_{H,t} \equiv p^N_H\left(\widehat{\psi}_t, t\right) = 1 - \mathcal{N}\left(\underline{\psi}; \widehat{\psi}_t + \sigma^H_{y\widehat{\psi}}, \sigma^2_{\widehat{\psi},t}\right) \tag{B29}$$

For the old economy

$$\boldsymbol{\sigma}_M = A_1(\tau)\boldsymbol{\sigma} + (S_{M,t} + S_{\pi,t})\,\widetilde{\boldsymbol{\sigma}}_\psi$$

where

$$S_{M,t} = \frac{-\frac{\partial p}{\partial \widehat{\psi}} G^{no}_t + \left((1-\gamma) A_2(\tau^{**}) + \frac{1}{\overline{p}} \frac{\partial \overline{p}}{\partial \widehat{\psi}}\right) G^{yes}}{(1 - p_t) G^{no}_t + p_t G^{yes}_t} \tag{B30}$$

and

$$\overline{p}_t \equiv \overline{p}\left(\widehat{\psi}_t, t\right) = 1 - \mathcal{N}\left(\underline{\psi}; \widehat{\psi}_t + (1-\gamma) A_2(\tau^{**}) \sigma^2_{\widehat{\psi},t}, \sigma^2_{\widehat{\psi},t}\right)$$

Q.E.D.

Proof of Proposition 6: Consider the old economy first. Rewrite the M/B of the old economy as

$$MB_t = \frac{G^{no}_t + p_t H_t}{\widetilde{G}^{no}_t + p_t \widetilde{H}_t}$$

where $H_t = G^{yes}_t - G^{no}_t$, and $\widetilde{H}_t = \widetilde{G}^{yes}_t - \widetilde{G}^{no}_t$. Given the closed form formulas for all the functions, we can compute the first derivative of MB_t with respect to the probability of adoption of the new technology p_t :

$$\frac{\partial MB_t}{\partial p_t} = \frac{H_t \widetilde{G}^{no}_t - G^{no}_t \widetilde{H}_t}{\left(\widetilde{G}^{no}_t + p_t \widetilde{H}_t\right)^2}$$

That is, the M/B increases in p_t if and only if $H_t \widetilde{G}^{no}_t > G^{no}_t \widetilde{H}_t$. Substituting the closed form expressions, we obtain the condition $h_{old} > 0$ where

$$h_{old} = -\widetilde{\kappa} + A_2(\tau^{**})\,\widehat{\psi}_t + \frac{1}{2}(1 - 2\gamma) A_2(\tau^{**})^2 \widehat{\sigma}^2_t \tag{B31}$$

$$- \log\left(\frac{1 - \mathcal{N}\left(\underline{\psi}; \widehat{\psi}_t - \gamma A_2(\tau^{**}) \sigma^2_{\widehat{\psi},t}, \sigma^2_{\widehat{\psi},t}\right)}{1 - \mathcal{N}\left(\underline{\psi}; \widehat{\psi}_t + (1-\gamma) A_2(\tau^{**}) \sigma^2_{\widehat{\psi},t}, \sigma^2_{\widehat{\psi},t}\right)}\right) \tag{B32}$$

Consider now the new economy

$$MB^N_t = \frac{K_t R^N_{L,t} + p_t J_t}{\widetilde{G}^{no}_t + p_t \widetilde{H}_t}$$

where $\overline{J}_t = (1-\kappa)^{-\gamma} K_t^N R_H^N - K_t R_L^N$. The first derivative with respect to p_t is

$$\frac{\partial MB_t^N}{\partial p_t} = \frac{\overline{J}_t \widetilde{G}_t^{no} - K_t R_{L,t}^N \widetilde{H}_t}{\left(\widetilde{G}_t^{no} + p_t \widetilde{H}_t\right)^2}$$

Once again, the M/B of the new economy increases in p_t if and only if $\overline{J}_t \widetilde{G}_t^{no} - K_t \widetilde{H}_t R_{L,t}^N > 0$. Substituting, we obtain the condition $h_{new} > 0$

$$h_{new} = -\gamma A_2(\tau^{**}) A_2(\tau) \widehat{\sigma}_t^2 - \log\left(\frac{\mathcal{N}\left(\underline{\psi}; \widehat{\psi}_t + \sigma_{y\widehat{\psi}}^L, \sigma_{\widehat{\psi},t}^2\right)}{\mathcal{N}\left(\underline{\psi}; \widehat{\psi}_t, \sigma_{\widehat{\psi},t}^2\right)}\right) \quad \text{(B33)}$$

$$- \log\left(\frac{1-\mathcal{N}\left(\underline{\psi}; \widehat{\psi}_t - \gamma A_2(\tau^{**}) \sigma_{\widehat{\psi},t}^2, \sigma_{\widehat{\psi},t}^2\right)}{1-\mathcal{N}\left(\underline{\psi}; \widehat{\psi}_t - \gamma A_2(\tau^{**}) \sigma_{\widehat{\psi},t}^2 + \sigma_{y\psi}^L, \sigma_{\widehat{\psi},t}^2\right)}\right) \quad \text{(B34)}$$

Proof of Proposition 7: Consider $\frac{M^N}{B^N} = \frac{\Phi^N}{\widetilde{\pi}}$, where Φ^N and $\widetilde{\pi}$ are defined appropriately. Then,

$$\frac{\partial\left(\frac{M^N}{B^N}\right)}{\partial \widehat{\psi}_t} = \frac{\widetilde{\pi}\partial\Phi^N/\partial\widehat{\psi}_t - \Phi^N \partial\widetilde{\pi}/\partial\widehat{\psi}_t}{\widetilde{\pi}^2} > 0$$

if and only if $S_{M,t}^N + S_{\pi,t} > 0$ where $S_{M,t}^N$, and $S_{\pi,t}$ are defined above. The probability of adoption as of time t^* is given by

$$p_{t^*} = \int_{f\left(\kappa,\gamma,\widehat{\sigma}_{t^*}^2;\tau^*\right)}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$$

where

$$f\left(\kappa,\gamma,\widehat{\sigma}_{t^*}^2;\tau^*\right) = -\log(1-\kappa)/A_2(\tau^{**}) \left(\frac{\left(\widehat{\sigma}_{t^*}^2\right)^{-1} + \left(\frac{\phi}{\sigma_{N,1}}\right)^2 (t^{**}-t^*)}{\widehat{\sigma}_{t^*}^2 \left(\frac{\phi}{\sigma_{N,1}}\right)^2 (t^{**}-t^*)}\right)^{\frac{1}{2}}$$

$$+ \frac{\frac{1}{2}(\gamma-1) A_2(\tau^{**})}{\left(\frac{\phi}{\sigma_{N,1}}\right)(t^{**}-t^*)^{\frac{1}{2}} \left(1 + \widehat{\sigma}_{t^*}^2 \left(\frac{\phi}{\sigma_{N,1}}\right)^2 (t^{**}-t^*)\right)^{\frac{1}{2}}}$$

Thus, p_t is small whenever $f\left(\kappa,\gamma,\widehat{\sigma}_{t^*}^2;\tau^*\right)$ is large. We can see that $f\left(\kappa,\gamma,\widehat{\sigma}_{t^*}^2;\tau^*\right)$ is large when κ is high, γ is high and, finally, when $\widehat{\sigma}_{t^*}^2$ is small (if $\kappa > 0$). (In addition, we can see that f is large when $(t^{**}-t^*)$ is small and T is large, the latter due to the increase in $A_2(\tau^{**}) = (T-t^{**}) - \left(1-e^{-\phi(T-t^{**})}\right)/\phi$). In all of these cases, the formulas for the various quantities in $S_{M,t}^N + S_{\pi,t}$ imply that the latter becomes positive. Q.E.D.

Proof of Lemma A6: Let

$$y_{t^{**}} = -\gamma b_{t^{**}} + b_{t^{**}}^N - \gamma A_1(t^{**};T)\rho_{t^{**}} + A_1(t^{**};T)\rho_{t^{**}}^N + (1-c_1) A_2(t^{**};T)\widehat{\psi}_{t^{**}}$$

The fact that $y_{t^{**}}$ and $\widehat{\psi}_{t^{**}}$ are jointly normally distributed stems from the linearity of all of the processes. To compute the means, variances and covariances, we can compute the joint moment generating function. That is, let $\alpha_1, \alpha_2 > 0$, and define

$$N\left(b_t, b_t^N, \rho_t, \rho_t^N, \widehat{\psi}_t, \widehat{\sigma}_t^2, t\right) = E_t\left[e^{\alpha_1 y_{t^{**}} + \alpha_2 \widehat{\psi}_{t^{**}}}\right]$$

where the processes of stochastic variables are given by (B7) - (B12) with $c = 0$. Let $\mathbf{x}_t = \left(b_t, b_t^N, \rho_t, \rho_t^N, \widehat{\psi}_t, \widehat{\sigma}_t^2\right)$, the Feynman-Kac theorem shows that N must satisfy the PDE

$$0 = \frac{\partial N}{\partial t} + \sum_i \frac{\partial N}{\partial x_i} E_t\left[dx\right] + \frac{1}{2}\sum_i \sum_j \frac{\partial^2 N}{\partial x_i \partial x_j} E_t\left[dx_i dx_j\right]$$

with the boundary condition $N\left(b_{t^{**}}, b_{t^{**}}^N, \rho_{t^{**}}, \rho_{t^{**}}^N, \widehat{\psi}_{t^{**}}, \widehat{\sigma}_{t^{**}}^2, t^{**}\right) = e^{\alpha_1 y_{t^{**}} + \alpha_2 \widehat{\psi}_{t^{**}}}$. It can be verified that the solution to the PDE is given by

$$N_t = e^{\alpha_1\left\{-\gamma b_t + b_t^N - \gamma C_1(t;T)\rho_t + C_1(t;T)\rho_t^N\right\} + \alpha_1 C_0(t;T) + \left\{(1-c_1)\alpha_1 C_2(t;T) + \alpha_2\right\}\widehat{\psi}_t + \alpha_1 C_3(t;T)\widehat{\sigma}_t^2}$$

where

$$\begin{aligned}
C_1\left(t;T\right) &= \frac{1-e^{-\phi(T-t)}}{\phi} = A_1\left(t;T\right) \\
C_2\left(t;T\right) &= A_2\left(t^{**};T\right) + \frac{1}{\left(1-c_1\right)} a\left(t\right) \\
\alpha_1 C_3\left(t;T\right) &= \widetilde{C}_3\left(t;T\right) \\
&= \frac{1}{2}\left(\left(1-c_1\right)\alpha_1 C_2 + \alpha_2\right)^2 - \frac{1}{2}\left(\left(1-c_1\right)\alpha_1 A_2\left(t^{**};T\right) + \alpha_2\right)^2
\end{aligned}$$

and

$$\begin{aligned}
\alpha_1 C_0\left(t;T\right) &= \widetilde{C}_0\left(t;T\right) \\
&= \alpha_1\left(1-\gamma\right)\overline{\rho} a\left(t\right) + \frac{1}{2}\left(\left(1-c_1\right)\alpha_1 A_2\left(t^{**};T\right) + \alpha_2\right)^2\left[\widehat{\sigma}_t^2 - \widehat{\sigma}_{t^{**}}^2\right] \\
&\quad + \alpha_1^2 \frac{1}{2}\left(\sigma^*\right)^2 a_2\left(t\right)
\end{aligned}$$

Above, $a\left(t\right)$, $a_2\left(t\right)$ and σ^* are given by (B23) - (B25). Rewrite $N_t = e^{g(\alpha_1, \alpha_2)}$ where

$$g\left(\alpha_1, \alpha_2\right) = \alpha_1\left\{-\gamma b_t + b_t^N - \gamma C_1\left(t;T\right)\rho_t + C_1\left(t;T\right)\rho_t^N\right\} + \widetilde{C}_0\left(t;T\right) + \left\{\left(1-c_1\right)\alpha_1 C_2\left(t;T\right) + \alpha_2\right\}\widehat{\psi}_t + \widetilde{C}_3\left(t;T\right)\widehat{\sigma}_t^2$$

Thus

$$\frac{\partial N}{\partial \alpha_1} = e^g\left\{\left(-\gamma b_t + b_t^N - \gamma C_1 \rho_t + C_1 \rho_t^N\right) + \frac{\partial \widetilde{C}_0}{\partial \alpha_1} + \left(1-c_1\right) C_2 \widehat{\psi}_t + \frac{\partial \widetilde{C}_3}{\partial \alpha_1}\widehat{\sigma}_t^2\right\}$$

We can use

$$\begin{aligned}
\frac{\partial \widetilde{C}_0}{\partial \alpha_1} &= \left(1-\gamma\right)\overline{\rho} a\left(t\right) + \left(\left(1-c_1\right)\alpha_1 A_2\left(t^{**};T\right) + \alpha_2\right)\left(1-c_1\right) A_2\left(t^{**};T\right)\left[\widehat{\sigma}_t^2 - \widehat{\sigma}_{t^{**}}^2\right] \\
&\quad + \alpha_1\left(\sigma^*\right)^2 a_2\left(t\right)
\end{aligned}$$

and

$$\frac{\partial \widetilde{C}_3}{\partial \alpha_1} = \left(\left(1-c_1\right)\alpha_1 C_2 + \alpha_2\right)\left(1-c_1\right)C_2 - \left(\left(1-c_1\right)\alpha_1 A_2\left(t^{**};T\right) + \alpha_2\right)\left(1-c_1\right)A_2\left(t^{**};T\right)$$

Thus

$$\lim_{\alpha_1,\alpha_2\to 0}\frac{\partial N}{\partial \alpha_1} = \mu_y = \left\{-\gamma b_t + b_t^N - \gamma C_1\rho_t + C_1\rho_t^N + \left(1-\gamma\right)\overline{\rho}a\left(t\right) + \left(1-c_1\right)C_2\widehat{\psi}_t\right\}$$

Similarly

$$\frac{\partial N}{\partial \alpha_2} = e^g\left\{\frac{\partial \widetilde{C}_0}{\partial \alpha_2} + \widehat{\psi}_t + \frac{\partial \widetilde{C}_3}{\partial \alpha_2}\widehat{\sigma}_t^2\right\}$$

Since

$$\begin{aligned}
\frac{\partial \widetilde{C}_0}{\partial \alpha_2} &= \left(\left(1-c_1\right)\alpha_1 A_2\left(t^{**};T\right) + \alpha_2\right)\left[\widehat{\sigma}_t^2 - \widehat{\sigma}_{t^{**}}^2\right] \\
\frac{\partial \widetilde{C}_3}{\partial \alpha_2} &= \left(\left(1-c_1\right)\alpha_1 C_2 + \alpha_2\right) - \left(\left(1-c_1\right)\alpha_1 A_2\left(t^{**};T\right) + \alpha_2\right)
\end{aligned}$$

we find

$$\lim_{\alpha_1,\alpha_2\to 0}\frac{\partial N}{\partial \alpha_2} = \mu_\psi = \widehat{\psi}_t$$

Turning to the second moments

$$\begin{aligned}
\frac{\partial^2 N}{\partial \alpha_1^2} &= e^g\left\{\left(-\gamma b_t + b_t^N - \gamma C_1\rho_t + C_1\rho_t^N\right) + \frac{\partial \widetilde{C}_0}{\partial \alpha_1} + \left(1-c_1\right)C_2\widehat{\psi}_t + \frac{\partial \widetilde{C}_3}{\partial \alpha_1}\widehat{\sigma}_t^2\right\}^2 \\
&\quad + e^g\left\{\frac{\partial^2 \widetilde{C}_0}{\partial \alpha_1^2} + \frac{\partial^2 \widetilde{C}_3}{\partial \alpha_1^2}\widehat{\sigma}_t^2\right\}
\end{aligned}$$

Since

$$\begin{aligned}
\frac{\partial^2 \widetilde{C}_0}{\partial \alpha_1^2} &= \left(1-c_1\right)^2 A_2\left(t^{**};T\right)^2\left[\widehat{\sigma}_t^2 - \widehat{\sigma}_{t^{**}}^2\right] + \left(\sigma^*\right)^2 a_2\left(t\right) \\
\frac{\partial^2 \widetilde{C}_3}{\partial \alpha_1^2} &= \left(\left(1-c_1\right)C_2\right)^2 - \left(\left(1-c_1\right)A_2\left(t^{**};T\right)\right)^2
\end{aligned}$$

we obtain

$$\begin{aligned}
\lim_{\alpha_1,\alpha_2\to 0}\frac{\partial^2 N}{\partial \alpha_1^2} &= \left\{-\gamma b_t + b_t^N - \gamma C_1\rho_t + C_1\rho_t^N + \left(1-\gamma\right)\overline{\rho}a\left(t\right) + \left(1-c_1\right)C_2\widehat{\psi}_t\right\}^2 \\
&\quad + \left(1-c_1\right)^2 A_2\left(t^{**};T\right)^2\left[\widehat{\sigma}_t^2 - \widehat{\sigma}_{t^{**}}^2\right] + \left(\sigma^*\right)^2 a_2\left(t\right) \\
&\quad + \left(\left(\left(1-c_1\right)C_2\right)^2 - \left(\left(1-c_1\right)A_2\left(t^{**};T\right)\right)^2\right)\widehat{\sigma}_t^2
\end{aligned}$$

Thus

$$\begin{aligned}
\sigma_y^2 &= \lim_{\alpha_1,\alpha_2\to 0}\frac{\partial^2 N}{\partial \alpha_1^2} - \left(\lim_{\alpha_1,\alpha_2\to 0}\frac{\partial N}{\partial \alpha_1}\right)^2 \\
&= \left(1-c_1\right)^2 A_2\left(t^{**};T\right)^2\left(\widehat{\sigma}_t^2 - \widehat{\sigma}_{t^{**}}^2\right) + a\left(\tau\right)^2\widehat{\sigma}_t^2 + 2A_2\left(t^{**};T\right)\left(1-c_1\right)a\left(\tau\right)\widehat{\sigma}_t^2 + \left(\sigma^*\right)^2 a_2\left(t\right)
\end{aligned}$$

Similarly,

$$\frac{\partial^2 N}{\partial \alpha_2^2} = e^g \left\{ \frac{\partial \widetilde{C}_0}{\partial \alpha_2} + \widehat{\psi}_t + \frac{\partial \widetilde{C}_3}{\partial \alpha_2} \widehat{\sigma}_t^2 \right\}^2 + e^g \left\{ \frac{\partial^2 \widetilde{C}_0}{\partial \alpha_2^2} + \frac{\partial^2 \widetilde{C}_3}{\partial \alpha_2^2} \widehat{\sigma}_t^2 \right\}$$

Since

$$\begin{aligned} \frac{\partial^2 \widetilde{C}_0}{\partial \alpha_2^2} &= \left[\widehat{\sigma}_t^2 - \widehat{\sigma}_{t^{**}}^2 \right] \\ \frac{\partial \widetilde{C}_3}{\partial \alpha_2} &= 0 \end{aligned}$$

we have

$$\lim_{\alpha_1, \alpha_2 \to 0} \frac{\partial^2 N}{\partial \alpha_2^2} = \widehat{\psi}_t^2 + \left[\widehat{\sigma}_t^2 - \widehat{\sigma}_{t^{**}}^2 \right]$$

and thus

$$\sigma_\psi^2 = \lim_{\alpha_1, \alpha_2 \to 0} \frac{\partial^2 N}{\partial \alpha_2^2} - \left(\lim_{\alpha_1, \alpha_2 \to 0} \frac{\partial N}{\partial \alpha_2} \right)^2 = \widehat{\sigma}_t^2 - \widehat{\sigma}_{t^{**}}^2$$

Finally

$$\begin{aligned} \frac{\partial^2 N}{\partial \alpha_2 \partial \alpha_1} &= e^g \left\{ \left(-\gamma b_t + b_t^N - \gamma C_1 \rho_t + C_1 \rho_t^N \right) + \frac{\partial \widetilde{C}_0}{\partial \alpha_1} + (1 - c_1) C_2 \widehat{\psi}_t + \frac{\partial \widetilde{C}_3}{\partial \alpha_1} \widehat{\sigma}_t^2 \right\} \left\{ \frac{\partial \widetilde{C}_0}{\partial \alpha_2} + \widehat{\psi}_t + \frac{\partial \widetilde{C}_3}{\partial \alpha_2} \widehat{\sigma}_t^2 \right\} \\ &\quad + e^g \left\{ \frac{\partial^2 \widetilde{C}_0}{\partial \alpha_2 \partial \alpha_1} + \frac{\partial^2 \widetilde{C}_3}{\partial \alpha_2 \partial \alpha_1} \widehat{\sigma}_t^2 \right\} \end{aligned}$$

Since

$$\begin{aligned} \frac{\partial^2 \widetilde{C}_0}{\partial \alpha_2 \partial \alpha_1} &= (1 - c_1) A_2 (t^{**}; T) \left[\widehat{\sigma}_t^2 - \widehat{\sigma}_{t^{**}}^2 \right] \\ \frac{\partial^2 \widetilde{C}_3}{\partial \alpha_2 \partial \alpha_1} &= (1 - c_1) (C_2 - A_2 (t^{**}; T)) \end{aligned}$$

we have

$$\begin{aligned} \lim_{\alpha_1, \alpha_2 \to 0} \frac{\partial^2 N}{\partial \alpha_2 \partial \alpha_1} &= \left\{ -\gamma b_t + b_t^N - \gamma C_1 \rho_t + C_1 \rho_t^N + (1 - \gamma) \overline{\rho} a(t) + (1 - c_1) C_2 \widehat{\psi}_t \right\} \left\{ \widehat{\psi}_t \right\} \\ &\quad + \left\{ (1 - c_1) A_2 (t^{**}; T) \left[\widehat{\sigma}_t^2 - \widehat{\sigma}_{t^{**}}^2 \right] + (1 - c_1) (C_2 - A_2 (t^{**}; T)) \widehat{\sigma}_t^2 \right\} \end{aligned}$$

implying

$$\begin{aligned} \sigma_{y,\psi} &= \left(\lim_{\alpha_1, \alpha_2 \to 0} \frac{\partial^2 N}{\partial \alpha_2 \partial \alpha_1} \right) - \left(\lim_{\alpha_1, \alpha_2 \to 0} \frac{\partial N}{\partial \alpha_1} \right) \left(\lim_{\alpha_1, \alpha_2 \to 0} \frac{\partial N}{\partial \alpha_2} \right) \\ &= (1 - c_1) A_2 (t^{**}; T) \left(\widehat{\sigma}_t^2 - \widehat{\sigma}_{t^{**}}^2 \right) + a(t) \widehat{\sigma}_t^2 \end{aligned}$$

Some additional algebra yields the formulas in Lemma A6. Q.E.D.