Technical Appendix for "The Wealth-Consumption Ratio"

A Pricing Aggregate Consumption and Labor Income

Environment Let $z_t \in Z$ be the aggregate state vector. We use z^t to denote the history of aggregate state realizations. Section 1.3 describes the dynamics of the aggregate state z_t of this economy, including the dynamics of aggregate consumption $C_t(z^t)$ and aggregate labor income $L_t(z^t)$.

We consider an economy that is populated by a continuum of heterogeneous agents, whose labor income is subject to idiosyncratic shocks. The idiosyncratic shocks are denoted by $\ell_t \in \mathcal{L}$, and we use ℓ^t to denote the history of these shocks. The household labor income process is given by:

$$\eta_t(\ell^t, z^t) = \widehat{\eta}_t(\ell^t, z_t) L_t(z^t).$$

Let $\Phi_t(z^t)$ denote the distribution of household histories ℓ^t conditional on being in aggregate node z^t. The labor income shares $\hat{\eta}$ aggregate to one:

$$\int \widehat{\eta}_t(\ell^t, z_t) d\Phi_t(z^t) = 1.$$

Trading in securities markets A non-zero measure of these households can trade bonds and stocks in securities markets that open every period. These households are in partition 1. We assume that the returns of these securities span Z. In other words, the payoff space is $R^{Z \times t}$ in each period t. Households in partition 2 can only trade one-period riskless discount bonds (a cash account). We use A^j to denote the menu of traded assets for households in segment $j \in \{1, 2\}$. However, none of these households can insure directly against idiosyncratic shocks ℓ_t to their labor income by selling a claim to their labor income or by trading contingent claims on these idiosyncratic shocks.

Law of One Price We assume free portfolio formation, at least for some households, and the law of one price. There exists a unique pricing kernel Λ_t in the payoff space. Since there is a non-zero measure of households that trade assets that span z_t, it only depends on the aggregate shocks z_t. Formally, this pricing kernel is the projection of any candidate pricing kernel on the space of traded payoffs $\underline{X}_t = \mathcal{R}^{Z \times t}$:

$$\frac{\Lambda_t}{\Lambda_{t-1}} = proj(M_t | \mathcal{R}^{Z \times t}).$$

We let P_t be the arbitrage-free price of an asset with payoffs $\{D_t^i\}$:

$$P_t^i = E_t \sum_{\tau=t}^{\infty} \frac{\Lambda_\tau}{\Lambda_t} D_\tau^i. \tag{21}$$

for any non-negative stochastic dividend process D_t^i that is measurable w.r.t z^t.

Household Problem We adopted the approach of Cuoco and He (2001): We let agents trade a full set of Arrow securities (contingent on both aggregate and idiosyncratic shock histories), but impose *measurability* restrictions on the positions in these securities.

After collecting their labor income and their payoffs from the Arrow securities, households buy consumption in spot markets and take Arrow positions $a_{t+1}(\ell^{t+1}, z^{t+1})$ in the securities markets subject to a standard budget constraint:

$$c_t + E_t\left[\frac{\Lambda_{t+1}}{\Lambda_t} a_t(\ell^{t+1}, z^{t+1})\right] + \sum_{i\in A^j} P_t^j s_{t+1}^j \leq \theta_t,$$

where s denotes the shares in a security j that is in the trading set of that agent. In the second term on the left-hand side, the expectations operator arises because we sum across all states of nature tomorrow and weight the price of each of the corresponding Arrow securities by the probability of that state arising. Wealth evolves according to:

$$\theta_{t+1} = a_t(\ell^{t+1}, z^{t+1}) + \eta_{t+1} + \sum_{i\in A^j}\left[P_{t+1}^j + D_{t+1}^j\right] s_t^j,$$

subject to a measurability constraint:

$$a_t(\ell^{t+1}, z^{t+1}) \textit{ is measurable w.r.t. } A_t^j(\ell^{t+1}, z^{t+1}),\ j \in \{1, 2\},$$

and subject to a generic borrowing or solvency constraint:

$$a_t(\ell^{t+1}, z^{t+1}) \geq B_t(\ell^t, z^t).$$

These measurability constraints limit the dependence of total household financial wealth on (z^{t+1}, ℓ^{t+1}). For example, for those households in partition 2 that only trade a risk-free bond, $A_t^2(\ell^{t+1}, z^{t+1}) = (\ell^t, z^t)$, because their net wealth can only depend on the history of aggregate and idiosyncratic states up until t. The households in partition 1, who do trade in stock and bond markets, can have net wealth that additionally depends on the aggregate state at time $t+1$: $A_t^1(\ell^{t+1}, z^{t+1}) = (\ell^t, z^{t+1})$.

Pricing of Household Human wealth In the absence of arbitrage opportunities, we can eliminate trade in actual securities (they are redundant), and the budget constraint reduces to:

$$c_t + E_t\left[\frac{\Lambda_{t+1}}{\Lambda_t} a_t(\ell^{t+1}, z^{t+1})\right] \leq a_{t-1}(\ell^t, z^t) + \eta_t.$$

By forward substitution of $a_t(\ell^{t+1}, z^{t+1})$ in the budget constraint, and by imposing the transversality condition on household net wealth:

$$\lim_{t\to\infty} \Lambda_t a_t(\ell^t, z^t) = 0,$$

it becomes apparent that the expression for financial wealth is :

$$\begin{aligned} a_{t-1}(\ell^t, z^t) &= E_t\left[\sum_{\tau=t}^{\infty} \frac{\Lambda_\tau}{\Lambda_t}\left(c_\tau(\ell^\tau, z^\tau) - \eta_\tau(\ell^\tau, z^\tau)\right)\right] \\ &= E_t\left[\sum_{\tau=t}^{\infty} \frac{\Lambda_\tau}{\Lambda_t} c_\tau(\ell^\tau, z^\tau)\right] - E_t\left[\sum_{\tau=t}^{\infty} \frac{\Lambda_\tau}{\Lambda_t} \eta_\tau(\ell^\tau, z^\tau)\right] \end{aligned}$$

The equation states that non-human wealth (on the left) equals the present discounted value of consumption minus the present discounted value of labor income (human wealth). The value of a claim to $c - y$ is *uniquely* pinned down, because the object on the left hand side is traded financial wealth.

Pricing of Aggregate Human Wealth Let Φ_0 denote the measure at time 0 over the history of idiosyncratic shocks. The (shadow) price of a claim to aggregate labor income at time 0 is given by the aggregation of the valuation of the household labor income streams:

$$\begin{aligned} & \int E_0\left[\sum_{t=0}^{\infty}\frac{\Lambda_t}{\Lambda_0}\left(\widehat{c}_t(\ell^t,z_t)C_t(z^t)-\widehat{\eta}_t(\ell^t,z_t)L_t(z^t)\right)\right]d\Phi_0 \\ = \quad & E_0\left[\sum_{t=0}^{\infty}\frac{\Lambda_t}{\Lambda_0}\int\left(\widehat{c}_t(\ell^t,z_t)d\Phi_t(z^t)C_t(z^t)-\widehat{\eta}_t(\ell^t,z_t)d\Phi_t(z^t)L_t(z^t)\right)\right], \\ = \quad & E_0\left[\sum_{t=0}^{\infty}\frac{\Lambda_t}{\Lambda_0}[C_t(z^t)-L_t(z^t)]\right], \end{aligned}$$

where we have used the fact that the pricing kernel Λ_t does not depend on the idiosyncratic shocks, the labor income shares integrate to one $\int\widehat{\eta}_t(\ell^t,z_t)d\Phi_t(z^t)=1$, and the consumption shares integrate to one $\int\widehat{c}_t(\ell^t,z_t)d\Phi_t(z^t)=1$, in which $\Phi_t(z^t)$ is the distribution of household histories ℓ^t conditional on being in aggregate node z^t.
Under the maintained assumption that the traded assets span aggregate uncertainty, this implies that aggregate human wealth is the present discounted value of aggregate labor income and that total wealth is the present discounted value of aggregate consumption, and the discounting is done with the projection of the SDF on the space of traded payoff space. Put differently, the discount factor is the same one that prices tradeable securities, such as stocks and bonds. This result follows directly from aggregating households' budget constraints. The result obtains despite the fact that human wealth is non-tradeable in this model, and therefore, markets are incomplete.

No Spanning If the traded payoffs do not span the aggregate shocks then the preceding argument still goes through for the projection of the candidate SDF on the space of traded payoffs:

$$\frac{\Lambda_t^*}{\Lambda_{t-1}^*}=proj\left(M_t|\underline{X}_t\right).$$

We can still price the aggregate consumption and labor income claims using Λ^*. The exact same reasoning applies. In this case, the part of non-traded payoffs that is orthogonal to the traded payoffs, may be priced:

$$\begin{aligned} E_t\left[\left(C_{t+1}-proj\left(C_{t+1}|\underline{X}_{t+1}\right)\right)\Lambda_{t+1}^*\right] &\neq 0, \\ E_t\left[\left(Y_{t+1}-proj\left(Y_{t+1}|\underline{X}_{t+1}\right)\right)\Lambda_{t+1}^*\right] &\neq 0, \end{aligned}$$

where we assume that $\underline{X}$ includes a constant so that the residuals are mean zero. In the main paper, we compute a good-deal bound on the consumption and labor income claims.

B Data Appendix

B.1 Macroeconomic Series

Labor income Our data are quarterly and span the period 1952.I-2006.IV. They are compiled from the most recent data available. Labor income is computed from NIPA Table 2.1 as wage and salary disbursements (line 3) + employer contributions for employee pension and insurance funds (line 7) + government social benefits to persons (line 17) - contributions for government social insurance (line 24) + employer contributions for government social insurance (line 8) - labor taxes. As in Lettau and Ludvigson (2001a), labor taxes are defined by imputing a share of personal current taxes (line 25) to labor income, with the share calculated as the ratio of wage and salary disbursements to the sum of wage and salary disbursements, proprietors' income (line 9), and rental income of persons with capital consumption adjustment (line 12), personal interest income (line 14) and personal dividend income (line 15). The series is seasonally-adjusted at annual rates (SAAR), and we divide it by 4. Because net worth of non-corporate business and owners' equity in farm business is part of financial wealth, it cannot also be part of human wealth. Consequently, labor income excludes proprietors' income.

Consumption Non-housing consumption consists of non-housing, non-durable consumption and non-housing durable consumption. Consumption data are taken from Table 2.3.5. from the Bureau of Economic Analysis' National Income and Product Accounts (BEA, NIPA). Non-housing, non-durable consumption is measured as the sum of non-durable goods (line 6) + services (line 13) - housing services (line14).

Non-housing durable consumption is unobserved and must be constructed. From the BEA, we observe durable *expenditures*. The value of the durables (Flow of Funds, see below) at the end of two consecutive quarters and the durable expenditures allows us to measure the implicit depreciation rate that entered in the Flow of Fund's calculation. We average that depreciation rate over the sample; it is δ=5.293% per quarter. We apply that depreciation rate to the value of the durable stock at the beginning of the current period (= measured as the end of the previous quarter) to get a time-series of this period's durable *consumption*.

We use housing services consumption (BEA, NIPA, Table 2.3.5, line 14) as the dividend stream from housing wealth. The BEA measures rent for renters and imputes a rent for owners. These series are SAAR, so we divide them by 4 to get quarterly values.

Total consumption is the sum of non-housing non-durable, non-housing durable, and housing consumption.

Population and deflation Throughout, we use the disposable personal income deflator from the BEA (Table 2.1, implied by lines 36 and 37) as well as the BEA's population series (line 38).

B.2 Financial Series

Stock market return We use value-weighted quarterly returns (NYSE, AMEX, and NASDAQ) from CRSP as our measure of the stock market return. In constructing the dividend-price ratio, we use the repurchase-yield adjustment advocated by Boudoukh, Michaely, Richardson, and Roberts (2004). We also add the dividends over the current and past three quarters, so as to obtain a price-dividend ratio that is comparable with an annual number.

Bond yields We use the nominal yield on a 3-month Treasury bill from Fama (CRSP file) as our measure of the risk-free rate. We also use the yield spread between a 5-year Treasury note and a 3-month Treasury bill as a return predictor. The 5-year yield is obtained from the Fama-Bliss data (CRSP file). The same Fama-Bliss yields of maturities 1-, 2-, 3-, 4-, and 5-years are used to form annual forward rates and to form 1-year excess returns in the Cochrane-Piazzesi excess bond return regression..

As additional yields to fit, we use the Fama-Bliss yields at the 1-year, and 3-year maturities. We also use yields on nominal bonds at maturities 10 and 20 years. We use yield data from the Federal Reserve Bank of Saint Louis (FRED II) for the latter, and construct the spread with the 5-year yield from FRED. The 10- and 20-year yields we use are the sum of the 5-year Fama-Bliss yield and the 10-5 and 20-5 yield spread from FRED. This is to adjust for any level differences in the 5-year yield between the two data sources. The 20-year yield data are missing in from 1987.I until 1993.III. The estimation can handle these missing observations because it minimizes the sum of squared differences between model-implied and observed yields, where the sum is only taken over available dates.

In order to plot the average yield curve in Figure 1, and only for this purpose, we also use the 7-5 year and the 30-5 year spread from FRED II. We add them to the 5-year yield from Fama-Bliss to form the 7-year and 30-year yield series. Since the 7-year yield data are missing from 1953.4-1969.6, we use spline interpolation (using the 1-, 2-, 5-, 10-, and 20-year yields) to fill in the missing data. The 30-year bond yield data are missing from 1953.4-1977.1 and from 2002.3-2006.1. We use the 20-year yield in those periods as a proxy. In the period where the 20-year yield is absent, we use the 30-year yield data in that period as a proxy. The resulting average 5-year yield is 6.11% per annum (straight from Fama-Bliss), the average 7-year yield is 6.25%, 10-year yield is 6.32%, 20-year is 6.51%, and the average 30-year yield is 6.45%.

Additional cross-sectional stock and bond returns We also use the 25 size and value equity portfolio returns from Kenneth French. We form log real quarterly returns.

C No-Arbitrage Model Details

C.1 Proof of Proposition 2

Proof. To find A_0^c and A_1^c, we need to solve the Euler equation for a claim to aggregate consumption. This Euler equation can either be thought of as the Euler equation that uses the nominal log SDF $m_{t+1}^{\$}$ to price the nominal total wealth return $\pi_{t+1} + r_{t+1}^c$ or the real log SDF $m_{t+1}^{\$} + \pi_{t+1}$ to price the real return r_{t+1}^c:

$$\begin{aligned}
1 &= E_t[\exp\{m_{t+1}^{\$} + \pi_{t+1} + r_{t+1}^c\}] \\
&= E_t[\exp\{-y_t^{\$}(1) - \frac{1}{2}L_t'L_t - L_t'\varepsilon_{t+1} + \pi_0 + e_\pi' z_{t+1} + \mu_c + e_{\Delta c}' z_{t+1} + A_0^c + A_1^{c\prime} z_{t+1} + \kappa_0^c - \kappa_1^c\left(A_0^c + A_1^{c\prime} z_t\right)\}] \\
&= \exp\{-y_0^{\$}(1) + \pi_0 - e_{yn}' z_t - \frac{1}{2}L_t'L_t + e_\pi'\Psi z_t + \kappa_0^c + (1-\kappa_1^c)A_0^c + \mu_c - \kappa_1^c A_1^{c\prime} z_t + \left(e_{\Delta c}' + A_1^{c\prime}\right)\Psi z_t\} \times \\
&\quad E_t\left[\exp\{-L_t'\varepsilon_{t+1} + (e_{\Delta c} + e_\pi + A_1^c)'\Sigma^{\frac{1}{2}}\varepsilon_{t+1}\}\right]
\end{aligned}$$

First, note that because of log-normality of ε_{t+1}, the last line equals:

$$\exp\left\{\frac{1}{2}\left(L_t'L_t + (e_{\Delta c} + e_\pi + A_1^c)'\Sigma(e_{\Delta c} + e_\pi + A_1^c)' - 2(e_{\Delta c} + e_\pi + A_1^c)'\Sigma^{\frac{1}{2}} L_t\right)\right\}$$

Substituting in for the expectation, as well as for the affine expression for L_t, we get

$$\begin{aligned}
1 &= \exp\{-y_0^{\$}(1) + \pi_0 - e_{yn}' z_t + \kappa_0^c + (1-\kappa_1^c)A_0^c + \mu_c - \kappa_1^c A_1^{c\prime} z_t + (e_{\Delta c} + e_\pi + A_1^c)'\Psi z_t\} \times \\
&\quad \exp\{\frac{1}{2}(e_{\Delta c} + e_\pi + A_1^c)'\Sigma(e_{\Delta c} + e_\pi + A_1^c) - (e_{\Delta c} + e_\pi + A_1^c)'\Sigma^{\frac{1}{2}}(L_0 + L_1 z_t)\}
\end{aligned}$$

Taking logs on both sides, an collecting the constant terms and the terms in z, we obtain the following:

$$\begin{aligned}
0 &= \{-y_0^{\$}(1) + \pi_0 + \kappa_0^c + (1-\kappa_1^c)A_0^c + \mu_c + \frac{1}{2}(e_{\Delta c} + e_\pi + A_1^c)'\Sigma(e_{\Delta c} + e_\pi + A_1^c) - (e_{\Delta c} + e_\pi + A_1^c)'\Sigma^{\frac{1}{2}} L_0\} + \\
&\quad \{-e_{yn}' - \kappa_1^c A_1^{c\prime} + (e_{\Delta c} + e_\pi + A_1^c)'\Psi - (e_{\Delta c} + e_\pi + A_1^c)'\Sigma^{\frac{1}{2}} L_1\} z_t
\end{aligned}$$

This equality needs to hold for all z_t. This is a system of $N+1$ equations in $N+1$ unknowns:

$$
\begin{aligned}
0 &= -y_0^{\$}(1)+\pi_0+\kappa_0^c+(1-\kappa_1^c)A_0^c+\mu_c+\frac{1}{2}(e_{\Delta c}+e_\pi+A_1^c)'\Sigma(e_{\Delta c}+e_\pi+A_1^c)-(e_{\Delta c}+e_\pi+A_1^c)'\Sigma^{\frac{1}{2}}L_0, && (22)\\
0 &= (e_{\Delta c}+e_\pi+A_1^c)'\Psi-\kappa_1^c A_1^{c\prime}-e_{yn}'-(e_{\Delta c}+e_\pi+A_1^c)'\Sigma^{\frac{1}{2}}L_1. && (23)
\end{aligned}
$$

The real short yield $y_t(1)$, or *risk-free rate*, satisfies $E_t[\exp\{m_{t+1}+y_t(1)\}]=1$. Solving out this Euler equation, we get:

$$
\begin{aligned}
y_t(1) &= y_t^{\$}(1)-E_t[\pi_{t+1}]-\frac{1}{2}e_\pi'\Sigma e_\pi+e_\pi'\Sigma^{\frac{1}{2}}L_t\\
&= y_0(1)+\left[e_{yn}'-e_\pi'\Psi+e_\pi'\Sigma^{\frac{1}{2}}L_1\right]z_t && (24)\\
y_0(1) &\equiv y_0^{\$}(1)-\pi_0-\frac{1}{2}e_\pi'\Sigma e_\pi+e_\pi'\Sigma^{\frac{1}{2}}L_0 && (25)
\end{aligned}
$$

The real short yield is the nominal short yield minus expected inflation minus a Jensen adjustment minus the inflation risk premium. Using the expression (25) for $y_0(1)$ in equation (22) delivers equation (6) in the main text. Likewise, using (24) in equation (23) delivers equation (7). □

Corollary 5. *The log price-dividend ratio on human wealth is a linear function of the (demeaned) state vector z_t*

$$pd_t^l = A_0^l + A_1^l z_t$$

where the following recursions pin down A_0^l and A_1^l:

$$
\begin{aligned}
0 &= \kappa_0^l+(1-\kappa_1^l)A_0^l+\mu_l-y_0(1)+\frac{1}{2}(e_{\Delta l}'+A_1^{l\prime})\Sigma(e_{\Delta l}+A_1^l)-(e_{\Delta l}'+A_1^{l\prime})\Sigma^{\frac{1}{2}}\left(L_0-\Sigma^{\frac{1}{2}\prime}e_\pi\right),\\
0 &= \left(e_{\Delta l}+e_\pi+A_1^l\right)'\Psi-\kappa_1^l A_1^{l\prime}-e_{yn}'-(e_{\Delta l}+e_\pi+A_1^l)'\Sigma^{\frac{1}{2}}L_1.
\end{aligned}
$$

The proof is identical to the proof of Proposition 2, and obtains by replacing μ_c by μ_l and the selector vector e_c by $e_{\Delta l}$. The linearization constants κ_0^l and κ_1^l relate to A_0^l through the analog of equation (3).

C.2 Campbell-Shiller Variance Decomposition

By iterating forward on the total wealth return equation (2), we can link the log wealth-consumption ratio at time t to expected future total wealth returns and consumption growth rates:

$$wc_t=\frac{\kappa_0^c}{\kappa_1^c-1}+\sum_{j=1}^{H}(\kappa_1^c)^{-j}\Delta c_{t+j}-\sum_{j=1}^{H}(\kappa_1^c)^{-j}r_{t+j}^c+(\kappa_1^c)^{-H}wc_{t+H}. \tag{26}$$

Because this expression holds both ex-ante and ex-post, one is allowed to add the expectation sign on the right-hand side. Imposing the transversality condition as $H\to\infty$ kills the last term, and delivers the familiar Campbell and Shiller (1988) decomposition for the price-dividend ratio of the consumption claim:

$$wc_t=\frac{\kappa_0^c}{\kappa_1^c-1}+E_t\left[\sum_{j=1}^{\infty}(\kappa_1^c)^{-j}\Delta c_{t+j}\right]-E_t\left[\sum_{j=1}^{\infty}(\kappa_1^c)^{-j}r_{t+j}\right]=\frac{\kappa_0^c}{\kappa_1^c-1}+\Delta c_t^H-r_t^H. \tag{27}$$

where the second equality follows from the definitions

$$\Delta c_t^H \equiv E_t\left[\sum_{j=1}^{\infty}(\kappa_1^c)^{-j}\,\Delta c_{t+j}\right] = e_c'\Psi(\kappa_1^c I - \Psi)^{-1}z_t, \tag{28}$$

$$r_t^H \equiv E_t\left[\sum_{j=1}^{\infty}(\kappa_1^c)^{-j}\,r_{t+j}\right] = \left[(e_c + A_1^c)'\Psi - \kappa_1^c A_1^{c\prime}\right](\kappa_1^c I - \Psi)^{-1}z_t, \tag{29}$$

where I is the $N \times N$ identity matrix. The first equation for the cash-flow component Δc_t^H follows from the VAR dynamics, while the second equation for the discount rate component r_t^H follows from Proposition 2 and the definition of the total wealth return equation (2).

Using expressions (29) and (28) and the log-linearity of the wealth-consumption ratio, we obtain analytical expressions for the following variance and covariance terms:

$$\begin{aligned}
V[wc_t] &= A_1^{c\prime}\Omega A_1^c && (30)\\
Cov\left[wc_t, \Delta c_t^H\right] &= A_1^{c\prime}\Omega(\kappa_1^c I - \Psi')^{-1}\Psi' e_c && (31)\\
Cov\left[wc_t, -r_t^H\right] &= A_1^{c\prime}\Omega\left[A_1^c - (\kappa_1^c I - \Psi)^{-1}\Psi' e_c'\right] && (32)\\
V\left[\Delta c_t^H\right] &= e_c'\Psi(\kappa_1^c I - \Psi)^{-1}\Omega(\kappa_1^c I - \Psi')^{-1}\Psi' e_c && (33)\\
V\left[r_t^H\right] &= \left[(e_c' + A_1^{c\prime})\Psi - \kappa_1^c A_1^{c\prime}\right](\kappa_1^c I - \Psi)^{-1}\Omega(\kappa_1^c I - \Psi')^{-1}\left[\Psi'(e_c + A_1^c) - \kappa_1^c A_1^c\right] && (34)\\
Cov\left[r_t^H, \Delta c_t^H\right] &= \left[(e_c' + A_1^{c\prime})\Psi - \kappa_1^c A_1^{c\prime}\right](\kappa_1^c I - \Psi)^{-1}\Omega(\kappa_1^c I - \Psi')^{-1}\Psi' e_c && (35)
\end{aligned}$$

where $\Omega = E[z_t' z_t]$ is the second moment matrix of the state z_t.

C.3 Nominal Term Structure

Proposition 6. *Nominal bond yields are affine in the state vector:*

$$y_t^{\$}(\tau) = -\frac{A^{\$}(\tau)}{\tau} - \frac{B^{\$}(\tau)'}{\tau}z_t,$$

where the coefficients $A^{\$}(\tau)$ and $B^{\$}(\tau)$ satisfy the following recursions

$$A^{\$}(\tau+1) = -y_0^{\$}(1) + A^{\$}(\tau) + \frac{1}{2}\left(B^{\$}(\tau)\right)'\Sigma\left(B^{\$}(\tau)\right) - \left(B^{\$}(\tau)\right)'\Sigma^{\frac{1}{2}}L_0, \tag{36}$$

$$\left(B^{\$}(\tau+1)\right)' = \left(B^{\$}(\tau)\right)'\Psi - e_{yn}' - \left(B^{\$}(\tau)\right)'\Sigma^{\frac{1}{2}}L_1, \tag{37}$$

initialized at $A^{\$}(0) = 0$ and $B^{\$}(0) = \mathbf{0}$.

Proof. We conjecture that the $t+1$-price of a τ-period bond is exponentially affine in the state

$$\log(P_{t+1}^{\$}(\tau)) = A^{\$}(\tau) + \left(B^{\$}(\tau)\right)' z_{t+1}$$

and solve for the coefficients $A^{\$}(\tau+1)$ and $B^{\$}(\tau+1)$ in the process of verifying this conjecture using the Euler equation:

$$
\begin{aligned}
P_t^{\$}(\tau+1) &= E_t[\exp\{m_{t+1}^{\$} + \log\left(P_{t+1}^{\$}(\tau)\right)\}] \\
&= E_t[\exp\{-y_t^{\$}(1) - \frac{1}{2}L_t'L_t - L_t'\varepsilon_{t+1} + A^{\$}(\tau) + \left(B^{\$}(\tau)\right)' z_{t+1}\}] \\
&= \exp\{-y_0^{\$}(1) - e_{yn}'z_t - \frac{1}{2}L_t'L_t + A^{\$}(\tau) + \left(B^{\$}(\tau)\right)' \Psi z_t\} \times \\
&\quad E_t\left[\exp\{-L_t'\varepsilon_{t+1} + \left(B^{\$}(\tau)\right)' \Sigma^{\frac{1}{2}}\varepsilon_{t+1}\}\right]
\end{aligned}
$$

We use the log-normality of ε_{t+1} and substitute for the affine expression for L_t to get:

$$
P_t^{\$}(\tau+1) = \exp\{-y_0^{\$}(1) - e_{yn}'z_t + A^{\$}(\tau) + \left(B^{\$}(\tau)\right)' \Psi z_t + \frac{1}{2}\left(B^{\$}(\tau)\right)' \Sigma \left(B^{\$}(\tau)\right) - \left(B^{\$}(\tau)\right)' \Sigma^{\frac{1}{2}}(L_0 + L_1 z_t)\}
$$

Taking logs and collecting terms, we obtain a linear equation for $\log(p_t(\tau+1))$:

$$
\log\left(P_t^{\$}(\tau+1)\right) = A^{\$}(\tau+1) + \left(B^{\$}(\tau+1)\right)' z_t,
$$

where $A^{\$}(\tau+1)$ satisfies (36) and $B^{\$}(\tau+1)$ satisfies (37). □

Real bond yields, $y_t(\tau)$, denoted without the \$ superscript, are affine as well with coefficients that follow similar recursions:

$$
\begin{aligned}
A(\tau+1) &= -y_0(1) + A(\tau) + \frac{1}{2}\left(B(\tau)\right)'\Sigma\left(B(\tau)\right) - \left(B(\tau)\right)'\Sigma^{\frac{1}{2}}\left(L_0 - \Sigma^{\frac{1}{2}\prime}e_\pi\right), \\
\left(B(\tau+1)\right)' &= \left(e_\pi + B(\tau)\right)'\Psi - e_{yn}' - \left(e_\pi + B(\tau)\right)'\Sigma^{\frac{1}{2}}L_1.
\end{aligned}
$$

For $\tau = 1$, we recover the expression for the risk-free rate in (24)-(25).

C.4 Dividend Strips

We define the return on equity conform the literature as $R_{t+1}^m = \frac{P_{t+1}^m + D_{t+1}^m}{P_t^m}$, where P_t^m is the end-of-period price on the equity market. A log-linearization delivers:

$$
r_{t+1}^m = \kappa_0^m + \Delta d_{t+1}^m + \kappa_1^m pd_{t+1}^m - pd_t^m. \tag{38}
$$

The unconditional mean stock return is $r_0^m = \kappa_0^m + (\kappa_1^m - 1)A_0^m + \mu_m$, where $A_0^m = E[pd_t^m]$ is the unconditional average log price-dividend ratio on equity and $\mu_m = E[\Delta d_t^m]$ is the unconditional mean dividend growth rate. The linearization constants κ_0^m and κ_1^m are different from the other wealth concepts because the timing of the return is different:

$$
\kappa_1^m = \frac{e^{A_0^m}}{e^{A_0^m}+1} < 1 \text{ and } \kappa_0^m = \log\left(e^{A_0^m}+1\right) - \frac{e^{A_0^m}}{e^{A_0^m}+1}A_0^m. \tag{39}
$$

Even though these constants arise from a linearization, we *define* log dividend growth so that the return equation holds exactly, given the CRSP series for $\{r_t^m, pd_t^m\}$. Our state vector z contains the (demeaned) return on the stock market, $r_{t+1}^m - r_0^m$, and the (demeaned) log price-dividend ratio $pd^m - A_0^m$. The definition of log equity returns allows us to back out dividend growth:

$$
\Delta d_{t+1}^m = \mu^m + \left[(e_{rm} - \kappa_1^m e_{pd})'\Psi + e_{pd}'\right] z_t + (e_{rm} - \kappa_1^m e_{pd})' \Sigma^{\frac{1}{2}}\varepsilon_{t+1}
$$

Proposition 7. *Log price-dividend ratios on dividend strips are affine in the state vector:*

$$p_t^d(\tau) = A^m(\tau) + B^{m\prime}(\tau) z_t,$$

where the coefficients $A^m(\tau)$ and $B^m(\tau)$ follow recursions

$$\begin{aligned}
A^m(\tau+1) &= A^m(\tau) + \mu_m - y_0(1) + \frac{1}{2}\left(e_{rm} - \kappa_1^m e_{pdm} + B^c(\tau)\right)' \Sigma \left(e_{rm} - \kappa_1^m e_{pdm} + B^m(\tau)\right) \\
&\quad - \left(e_{rm} - \kappa_1^m e_{pdm} + B^m(\tau)\right)' \Sigma^{\frac{1}{2}} \left(L_0 - \Sigma^{\frac{1}{2}} e_\pi\right), \\
B^m(\tau+1)' &= \left(e_{rm} - \kappa_1^m e_{pdm} + e_\pi + B^m(\tau)\right)' \Psi + e_{pdm}' - e_{yn}' - \left(e_{rm} - \kappa_1^m e_{pdm} + e_\pi + B^m(\tau)\right)' \Sigma^{\frac{1}{2}} L_1,
\end{aligned}$$

initialized at $A^m(0) = 0$ and $B^m(0) = 0$.

Proof. We conjecture that the log $t+1$-price of a τ-period strip, scaled by the dividend in period $t+1$, is affine in the state

$$p_{t+1}^d(\tau) = \log\left(P_{t+1}^d(\tau)\right) = A^m(\tau) + B^m(\tau)' z_{t+1}$$

and solve for the coefficients $A^m(\tau+1)$ and $B^m(\tau+1)$ in the process of verifying this conjecture using the Euler equation:

$$\begin{aligned}
P_t^d(\tau+1) &= E_t[\exp\{m_{t+1}^\$ + \pi_{t+1} + \Delta d_{t+1}^m + \log\left(p_{t+1}^m(\tau)\right)\}] \\
&= E_t[\exp\{-y_t^\$(1) - \frac{1}{2} L_t' L_t - L_t' \varepsilon_{t+1} + \pi_0 + e_\pi' z_{t+1} + \Delta d_{t+1}^m + A^m(\tau) + B^m(\tau)' z_{t+1}\}] \\
&= \exp\{-y_0^\$(1) - e_{yn}' z_t - \frac{1}{2} L_t' L_t + \pi_0 + e_\pi' \Psi z_t + \mu_m + \left[(e_{rm} - \kappa_1^m e_{pd})' \Psi + e_{pd}'\right] z_t + A^m(\tau) + B^m(\tau)' \Psi z_t\} \times \\
&\quad E_t\left[\exp\{-L_t' \varepsilon_{t+1} + \left(e_{rm} - \kappa_1^m e_{pd} + e_\pi + B^m(\tau)\right)' \Sigma^{\frac{1}{2}} \varepsilon_{t+1}\right]
\end{aligned}$$

We use the log-normality of ε_{t+1} and substitute for the affine expression for L_t to get:

$$\begin{aligned}
P_t^d(\tau+1) &= \exp\{-y_0^\$(1) - e_{yn}' z_t + \pi_0 + \mu_m + A^m(\tau) + \left[\left(e_{rm} - \kappa_1^m e_{pd} + e_\pi + B^m(\tau)\right)' \Psi + e_{pd}'\right] z_t + \\
&\quad \frac{1}{2}\left(e_{rm} - \kappa_1^m e_{pd} + e_\pi + B^m(\tau)\right)' \Sigma \left(e_{rm} - \kappa_1^m e_{pd} + e_\pi + B^m(\tau)\right) - \left(e_{rm} - \kappa_1^m e_{pd} + e_\pi + B^m(\tau)\right)' \Sigma^{\frac{1}{2}} (L_0 + L_1 z_t)\}
\end{aligned}$$

Taking logs and collecting terms, we obtain a log-linear expression for $p_t^m(\tau+1)$:

$$p_t^d(\tau+1) = A^m(\tau+1) + B^m(\tau+1)' z_t,$$

where

$$\begin{aligned}
A^m(\tau+1) &= A^m(\tau) + \mu_m - y_0^\$(1) + \pi_0 + \frac{1}{2}\left(e_{rm} - \kappa_1^m e_{pd} + e_\pi + B^c(\tau)\right)' \Sigma \left(e_{rm} - \kappa_1^m e_{pd} + e_\pi + B^m(\tau)\right) \\
&\quad - \left(e_{rm} - \kappa_1^m e_{pd} + e_\pi + B^m(\tau)\right)' \Sigma^{\frac{1}{2}} L_0, \\
B^m(\tau+1)' &= \left(e_{rm} - \kappa_1^m e_{pd} + e_\pi + B^m(\tau)\right)' \Psi + e_{pd}' - e_{yn}' - \left(e_{rm} - \kappa_1^m e_{pd} + e_\pi + B^m(\tau)\right)' \Sigma^{\frac{1}{2}} L_1
\end{aligned}$$

We recover the recursions of Proposition 7 after substituting out the expressions for the nominal yields using equations (24) and (25). □

C.5 Consumption Strips

Proposition 8. *Log price-dividend ratios on consumption strips are affine in the state vector:*

$$p_t^c(\tau) = A^c(\tau) + B^{c\prime}(\tau) z_t,$$

where the coefficients $A^c(\tau)$ and $B^c(\tau)$ follow recursions:

$$\begin{aligned}
A^c(\tau+1) &= A^c(\tau) + \mu_c - y_0(1) + \frac{1}{2}\left(e_c + B^c(\tau)\right)' \Sigma \left(e_c + B^c(\tau)\right) - \left(e_c + B^c(\tau)\right)' \Sigma^{\frac{1}{2}} \left(L_0 - \Sigma^{\frac{1}{2}} e_\pi\right), \\
B^c(\tau+1)' &= \left(e_c + e_\pi + B^c(\tau)\right)' \Psi - e_{yn}' - \left(e_c + e_\pi + B^c(\tau)\right)' \Sigma^{\frac{1}{2}} L_1.
\end{aligned}$$

initialized at $A^c(0) = 0$ and $B^c(0) = 0$.

Proof. We conjecture that the $t+1$-price of a τ-period strip is exponentially affine in the state

$$p_{t+1}^c(\tau) = A^c(\tau) + B^c(\tau)' z_{t+1}$$

and solve for the coefficients $A^c(\tau+1)$ and $B^c(\tau+1)$ in the process of verifying this conjecture using the Euler equation:

$$\begin{aligned}
P_t^c(\tau+1) &= E_t[\exp\{m_{t+1}^\$ + \pi_{t+1} + \Delta c_{t+1} + \log\left(p_{t+1}^c(\tau)\right)\}] \\
&= E_t[\exp\{-y_t^\$(1) - \frac{1}{2} L_t' L_t - L_t' \varepsilon_{t+1} + \pi_0 + e_\pi' z_{t+1} + \Delta c_{t+1} + A^c(\tau) + B^c(\tau)' z_{t+1}\}] \\
&= \exp\{-y_0^\$(1) - e_{yn}' z_t - \frac{1}{2} L_t' L_t + \pi_0 + e_\pi' \Psi z_t + \mu_c + e_{\Delta c}' \Psi z_t + A^c(\tau) + B^c(\tau)' \Psi z_t\} \times \\
&\quad E_t\left[\exp\{-L_t' \varepsilon_{t+1} + (e_{\Delta c} + e_\pi + B^c(\tau))' \Sigma^{\frac{1}{2}} \varepsilon_{t+1}\right]
\end{aligned}$$

We use the log-normality of ε_{t+1} and substitute for the affine expression for L_t to get:

$$\begin{aligned}
P_t^c(\tau+1) &= \exp\{-y_0^\$(1) - e_{yn}' z_t + \pi_0 + \mu_c + A^c(\tau) + \left(e_{\Delta c} + e_\pi + B^c(\tau)\right)' \Psi z_t + \frac{1}{2}\left(e_{\Delta c} + e_\pi + B^c(\tau)\right)' \Sigma \left(e_{\Delta c} + e_\pi + B^c(\tau)\right) \\
&\quad - \left(e_{\Delta c} + e_\pi + B^c(\tau)\right)' \Sigma^{\frac{1}{2}} \left(L_0 + L_1 z_t\right)\}
\end{aligned}$$

Taking logs and collecting terms, we obtain a log-linear expression for $p_t^c(\tau+1)$:

$$p_t^c(\tau+1) = A^c(\tau+1) + B^c(\tau+1)' z_t,$$

where

$$\begin{aligned}
A^c(\tau+1) &= A^c(\tau) + \mu_c - y_0^\$(1) + \pi_0 + \frac{1}{2}\left(e_{\Delta c} + e_\pi + B^c(\tau)\right)' \Sigma \left(e_{\Delta c} + e_\pi + B^c(\tau)\right) - \left(e_{\Delta c} + e_\pi + B^c(\tau)\right)' \Sigma^{\frac{1}{2}} L_0, \\
B^c(\tau+1)' &= \left(e_{\Delta c} + e_\pi + B^c(\tau)\right)' \Psi - e_{yn}' - \left(e_{\Delta c} + e_\pi + B^c(\tau)\right)' \Sigma^{\frac{1}{2}} L_1
\end{aligned}$$

We recover the recursions of Proposition 8 after substituting out the expressions for the nominal yields using equations (24) and (25). □

We decompose the yield on a τ-period consumption strip from Proposition 8, $y_t^c(\tau) = -p_t^c(\tau)/\tau$, into the yield on a τ-period real bond adjusted for consumption growth plus the yield on the consumption cash-flow risk security

$y_t^{ccr}(\tau)$:

$$y_t^c(\tau) = (y_t(\tau) - \mu_c) + y_t^{ccr}(\tau).$$

The former can be thought of as the period-τ coupon yield on a real perpetuity with cash-flows that grow at a deterministic rate μ_c, while the latter captures the cash-flow risk in the consumption claim. We have that $y_t^{ccr}(\tau) = -p_t^{ccr}(\tau)/\tau$. Since the log price-dividend ratio of the consumption strips and the log real bond prices are both affine, so is the log price-dividend ratio of the consumption cash-flow risk security: $\log p_t^{ccr}(\tau) = A^{ccr}(\tau) + B^{ccr}(\tau) z_t$. It is easy to show that its coefficients follow the recursions:

$$\begin{aligned}
A^{ccr}(\tau+1) &= A^{ccr}(\tau) + \frac{1}{2}\left(e_c + B^{ccr}(\tau)\right)' \Sigma \left(e_c + B^{ccr}(\tau)\right) + \left(e_c + B^{ccr}(\tau)\right)' \Sigma B(\tau) \\
&- \left(e_c + B^{ccr}(\tau)\right)' \Sigma^{\frac{1}{2}} \left(L_0 - \Sigma^{\frac{1}{2}} e_\pi\right), \\
B^{ccr}(\tau+1)' &= \left(e_c + B^{ccr}(\tau)\right)' \Psi - \left(e_c + B^{ccr}(\tau)\right)' \Sigma^{\frac{1}{2}} L_1.
\end{aligned}$$

C.6 Point Estimates

Below, we report the point estimates for the VAR companion matrix Ψ, the Cholesky decomposition of the covariance matrix $\Sigma^{.5}$ (multiplied by 100), and the market price of risk parameters L_0 and L_1 for our benchmark specification. We recall that the market price of risk parameter matrix L_1 pre-multiplies the state z_t, which has a (non-standardized) covariance matrix Ω.

$$\Psi = \begin{bmatrix}
\mathbf{0.3053} & \mathbf{1.2456} & \mathbf{-0.4565} & \mathbf{3.3037} & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.0035 & \mathbf{0.8965} & \mathbf{0.1022} & 0.1590 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbf{-0.0371} & \mathbf{0.1690} & \mathbf{0.7312} & 0.0930 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.0086 & 0.0289 & -0.0424 & \mathbf{0.6973} & 0 & 0 & 0 & 0 & 0 & 0 \\
0.3201 & -2.3434 & -0.9365 & -2.8085 & \mathbf{0.9131} & -0.0230 & 0 & 0 & 0 & 0 \\
0.2257 & -1.8392 & -1.5961 & -0.9984 & \mathbf{-0.0890} & 0.0429 & 0 & 0 & 0 & 0 \\
0.0139 & 0.0466 & -0.5851 & 0.6283 & -0.0105 & \mathbf{0.0708} & 0 & 0 & 0 & 0 \\
0.1506 & -0.2318 & -0.6371 & -0.0617 & -0.0130 & 0.0769 & 0 & 0 & 0 & 0 \\
0.0251 & -0.0202 & -0.0557 & 0.0350 & 0.0006 & -0.0020 & -0.0002 & 0.0068 & \mathbf{0.3014} & 0 \\
\mathbf{0.1032} & -0.2691 & 0.0091 & -0.6800 & 0.0016 & -0.0064 & 0.0495 & -0.0207 & \mathbf{0.5887} & -0.1390
\end{bmatrix}$$

$$\Sigma^{.5} \times 100 = \begin{bmatrix}
1.4427 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.0714 & 0.2331 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.0519 & 0.0352 & 0.2998 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.0838 & -0.1071 & 0.0136 & 0.1012 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.7999 & -0.6542 & -1.6896 & -0.3572 & 7.8117 & 0 & 0 & 0 & 0 & 0 \\
0.6247 & -0.6375 & -1.8040 & -0.4144 & 7.3491 & 2.5131 & 0 & 0 & 0 & 0 \\
0.4139 & -0.1845 & -0.4411 & -0.1536 & 1.7789 & 0.5328 & 2.5774 & 0 & 0 & 0 \\
1.0101 & -0.4506 & -0.6907 & -0.1438 & 2.2836 & 0.8748 & 3.8193 & 4.7776 & 0 & 0 \\
0.0292 & -0.0569 & 0.0166 & 0.0106 & 0.0801 & 0.0593 & 0.1280 & -0.0122 & 0.3631 & 0 \\
0.1751 & -0.0484 & -0.0046 & -0.0113 & 0.1342 & -0.0335 & 0.2257 & 0.2664 & 0.3247 & 0.6503
\end{bmatrix}$$

$$L_0' = \left[\begin{array}{cccccccccc} 0 & \mathbf{-0.1781} & 0 & \mathbf{0.0309} & 0 & \mathbf{0.6601} & -0.0712 & 0.1225 & 0 & 0 \end{array} \right]$$

$$L_1 = \left[\begin{array}{cccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{4.1229} & \mathbf{-70.9093} & \mathbf{55.3549} & \mathbf{-96.4021} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{-49.3448} & \mathbf{131.6779} & \mathbf{-48.3828} & \mathbf{249.5507} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.3578 & -101.5409 & -29.1252 & -17.9823 & \mathbf{-3.5997} & 1.7053 & 0 & 0 & 0 & 0 \\ -3.7005 & -5.7382 & 12.0120 & 41.0278 & 0.4508 & \mathbf{2.3711} & 0 & 0 & 0 & 0 \\ 4.2683 & -1.2674 & 1.0349 & -29.7221 & 0.1894 & -0.5990 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

We compute OLS standard errors for the elements of Ψ and report the coefficients with a t-stat greater than 1.98 in bold. Bootstrap standard errors on the market price of risk parameters are available upon request. They are derived as part of the method explained in Appendix C.7. Market price of risk estimates with bootstrap t-stats above 1.98 are indicated in bold.

C.7 Bootstrap Standard Errors

We obtain standard errors on the moments of the estimated wealth-consumption ratio by bootstrap. More precisely, we conduct two bootstrap exercises leading to two sets of standard errors. In each exercise, we draw with replacement from the VAR innovations ε_t. We draw row-by-row in order to preserve the cross-correlation structure between the state innovations (Step 1). Given the point estimates for Ψ and Σ as well as the mean vector μ, we recursively reconstruct the state vector (Step 2). We then re-estimate the mean vector, companion matrix, and innovation covariance matrix (Step 3). With the new state vector and the new VAR parameters in hand, we re-estimate the market price of risk parameters in L_0 and L_1 (Step 4). Just as in the main estimation, we use 2500 quarters to approximate the infinite-horizon sums in the strip price-dividend ratio calculations. We limit the estimation in Step 4 to 500 function evaluations for computational reasons. In some of the bootstrap iterations, the optimization in Step 4 does not find a feasible solution. This happens, for example when no parameter choices keep the human wealth share less than hundred percent or the consumption or labor income claim finite. We discard these bootstrap iterations. These new market price of risk parameters deliver a new wealth-consumption ratio time series (Step 5). With the bootstrap time series for consumption growth and the wealth-consumption ratio, we can form all the moments in Table 1. We repeat this procedure 1,000 times and report the standard deviation across the bootstrap iterations. We conduct two variations on the above algorithm. The first set of standard errors are reported in parentheses, the second set are in brackets in Table 1. Each bootstrap exercise takes about 12 hours to compute on an 8-processor computer.

In the first exercise, we only consider sampling uncertainty in the last four elements of the state: the two factor mimicking portfolios, consumption growth, and labor income growth. We assume that all the other variables are observed without error. The idea is that national account aggregates are measured much less precisely than traded stocks and bonds. This procedure takes into account sampling uncertainty in consumption growth and its correlations with yields and with the aggregate stock market. Given our goal of obtaining standard errors around the moments of the wealth-consumption ratio, this seems like a natural first exercise. The second column of Table 4 reports the standard errors from this bootstrap exercise in parentheses. For completeness, it also reports the mean across bootstrap iterations.

In a second estimation exercise, we also consider sampling uncertainty in the first six state variables (yields and stock prices). Redrawing the yields that enter in the state space (the 1-quarter yield and the 20-1 quarter yield spread) requires also redrawing the additional yields that are used in estimation (the 4-, 12-, 40-, and 80-quarter yields) and in the formation of the Cochrane-Piazzesi factor (the 4-, 8-, 12-, and 16- quarter yields). Otherwise, the bootstrapped time-series for the yields in the state space would be disconnected from the other yields. For this second exercise, we augment the VAR with the following yield spreads: 4-20, 8-20, 12-20, 16-20, 40-20, and 80-20 quarter yield spreads. We let these spreads depend on their own lag and on the lagged 1-quarter yield. Additional dependence on the lagged 20-1 quarter yield makes little difference. In Step 1, we draw from the yield spreads-augmented VAR innovations. This allows us to take into account the cross-dependencies between all the yields in the yield curve. In addition to recursively rebuilding the state variables in Step 2, we also rebuild the six yield spreads. With the bootstrapped yields, we reconstruct the forward rates, one-year excess bond returns, re-estimate the excess bond return regression, and re-construct the Cochrane-Piazzesi factor. Steps 3 through 5 are the same as in the first exercise. One additional complication arises because the bootstrapped yields often turn negative for one or more periods. Since negative nominal yields never happen in the data and make no economic sense, we discard these bootstrap iterations. We redraw from the VAR innovations until we have 1,000 bootstrap samples with strictly positive yields at all maturities. This is akin to a rejection-sampling procedure. One drawback is that there is an upward bias in the yield curve. The average one-quarter yield is 5.13% per annum in our data sample, while it is 5.93% in our bootstrap sample. This translates in a small downward bias in the average wealth consumption ratio: the average wealth-consumption ratio is 5.86 in the data and 5.69 in the bootstrap. The third column of Table 4 reports the standard errors from this bootstrap exercise in parentheses. As expected, the standard errors from the second bootstrap exercise are somewhat bigger. However, their difference is small. We report the more conservative standard errors in the main text.

D The Long-Run Risk Model

D.1 Preferences

Let $V_t(C_t)$ denote the utility derived from consuming C_t, then the value function of the representative agent takes the following recursive form:

$$V_t(C_t) = \left[(1-\beta)C_t^{1-\rho} + \beta(\mathcal{R}_t V_{t+1})^{1-\rho}\right]^{\frac{1}{1-\rho}}, \tag{40}$$

where the risk-adjusted expectation operator is defined as:

$$\mathcal{R}_t V_{t+1} = \left(E_t V_{t+1}^{1-\alpha}\right)^{\frac{1}{1-\alpha}}.$$

For these preferences, α governs risk aversion and ρ governs the willingness to substitute consumption inter-temporally. It is the inverse of the inter-temporal elasticity of substitution. In the special case where $\rho = \alpha$, they collapse to the standard power utility preferences, used in Breeden (1979) and Lucas (1978). Epstein and Zin

Table 4: Bootstrap Standard Errors

This table displays bootstrap standard errors on the unconditional moments of the log wealth-consumption ratio wc, its first difference Δwc, and the log total wealth return r^c. The last but one row reports the time-series average of the conditional consumption risk premium, $E[E_t[r_t^{c,e}]]$, where $r^{c,e}$ denotes the expected log return on total wealth in excess of the risk-free rate and corrected for a Jensen term. The first column repeats the point estimates from the main text (last column of Table 1). The second and third columns report the results from two bootstrap exercises, described above. The table reports the mean and standard deviation (in parentheses) across 1,000 bootstrap iterations.

Moments	Point Estimate	Bootstrap 1	Bootstrap 2
$Std[wc]$	17.24%	16.26%	15.24%
(s.e.)		(3.39)	(4.30)
$AC(1)[wc]$	.96	.95	.93
(s.e.)		(.00)	(.03)
$AC(4)[wc]$	.85	.83	.74
(s.e.)		(.01)	(.08)
$Std[\Delta wc]$	4.86%	4.86%	5.07%
(s.e.)		(0.98)	(1.16)
$Std[\Delta c]$	.44%	.44%	.44%
(s.e.)		(.03)	(.03)
$Corr[\Delta c, \Delta wc]$	.11	.02	.12
(s.e.)		(.06)	(.06)
$Std[r^c]$	4.94%	4.90%	5.16%
(s.e.)		(2.21)	(1.16)
$Corr[r^c, \Delta c]$	.19	.12	.21
(s.e.)		(.07)	(.07)
$E[E_t[r_t^{c,e}]]$	.54%	.46%	0.53%
(s.e.)		(.11)	(.16)
$E[wc]$	5.86	6.29	5.69
(s.e.)		(.48)	(.49)

(1989) show that the stochastic discount factor can be written as:

$$M_{t+1} = \beta \left(\frac{C_{t+1}}{C_t}\right)^{-\rho} \left(\frac{V_{t+1}}{\mathcal{R}_t V_{t+1}}\right)^{\rho-\alpha} \tag{41}$$

The next proposition shows that the ability to write the SDF in the long-run risk model as a function of consumption growth and the wealth-consumption ratio is general. It does not depend on the linearization of returns, nor on the assumptions on the stochastic process for consumption growth (see Hansen, Heaton, Lee, and Roussanov (2008)).

Proposition 9. *The log SDF in the non-linear version of the long-run risk model can be stated as*

$$m_{t+1} = \frac{1-\alpha}{1-\rho}\log\beta - \alpha\Delta c_{t+1} + \frac{\rho-\alpha}{1-\rho}\log\left(\frac{e^{wc_{t+1}}}{e^{wc_t}-1}\right) \tag{42}$$

Proof. We start from the value function definition in equation (40) and raise both sides to the power $1-\rho$, and subsequently divide through by $(1-\beta)C_t^{-\rho}$ to obtain:

$$\frac{V_t^{1-\rho}}{(1-\beta)C_t^{-\rho}} = C_t + \beta\frac{\left(E_t V_{t+1}^{1-\alpha}\right)^{\frac{1-\rho}{1-\alpha}}}{(1-\beta)C_t^{-\rho}} \tag{43}$$

Some algebra and the definition of the risk-adjusted expectation operator imply that

$$E_t(V_{t+1}^{1-\alpha})^{\frac{1-\rho}{1-\alpha}} = E_t(V_{t+1}^{1-\alpha})^{1-\frac{\rho-\alpha}{1-\alpha}} = \frac{E_t(V_{t+1}^{1-\alpha})}{E_t(V_{t+1}^{1-\alpha})^{\frac{\rho-\alpha}{1-\alpha}}} = \frac{E_t(V_{t+1}^{1-\alpha})}{(\mathcal{R}_t V_{t+1})^{\rho-\alpha}} = E_t\left[\frac{V_{t+1}^{\rho-\alpha}V_{t+1}^{1-\rho}}{(\mathcal{R}_t V_{t+1})^{\rho-\alpha}}\right]$$

The left-hand side shows up in equation (43), and we substitute it for the right-hand side of the equation above. Multiplying and dividing inside the expectation operator by $C_{t+1}^{-\rho}$, we get:

$$\frac{V_t^{1-\rho}}{(1-\beta)C_t^{-\rho}} = C_t + E_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\rho}\left(\frac{V_{t+1}}{\mathcal{R}_t V_{t+1}}\right)^{\rho-\alpha}\frac{V_{t+1}^{1-\rho}}{(1-\beta)C_{t+1}^{-\rho}}\right]$$

Note that the first three terms inside the expectation are equal to the stochastic discount factor in equation (41). This is a no-arbitrage asset pricing equation of an asset with dividend equal to aggregate consumption. The price of this asset is W_t. Hence,

$$W_t = C_t + E_t\left[M_{t+1}W_{t+1}\right] \quad \text{and} \quad W_t = \frac{V_t^{1-\rho}}{(1-\beta)C_t^{-\rho}}. \tag{44}$$

This equation, together with $E[M_{t+1}R_{t+1}^c] = 1$ delivers the return on the total wealth portfolio:

$$R_{t+1}^c = \frac{W_{t+1}}{(W_t - C_t)} = \frac{\frac{V_{t+1}^{1-\rho}}{(1-\beta)C_{t+1}^{-\rho}}}{\frac{V_t^{1-\rho}}{(1-\beta)C_t^{-\rho}} - C_t} = \frac{\frac{V_{t+1}^{1-\rho}}{(1-\beta)C_{t+1}^{-\rho}}}{\frac{\beta}{(1-\beta)C_t^{-\rho}}(\mathcal{R}_t V_{t+1})^{1-\rho}} = \beta^{-1}\left(\frac{C_{t+1}}{C_t}\right)^{\rho}\left(\frac{V_{t+1}}{\mathcal{R}_t V_{t+1}}\right)^{1-\rho}, \tag{45}$$

where the first equality is the return definition, the second one follows from the definition of W_t, and the third equality from the Bellman equation (40).

One then rearranges this equation (after raising both sides to the power $\frac{\rho-\alpha}{1-\rho}$) to get a relationship between the change in continuation values and the total wealth return:

$$\left(\frac{V_{t+1}}{\mathcal{R}_t V_{t+1}}\right)^{\rho-\alpha} = \beta^{\frac{\rho-\alpha}{1-\rho}}\left(\frac{C_{t+1}}{C_t}\right)^{-\rho\frac{\rho-\alpha}{1-\rho}}\left(R_{t+1}^c\right)^{\frac{\rho-\alpha}{1-\rho}}.$$

Finally, we substitute it into the stochastic discount factor expression (41) to obtain an expression that depends only on consumption growth and the return to the wealth portfolio:

$$M_{t+1} = \beta^{\frac{1-\alpha}{1-\rho}}\left(\frac{C_{t+1}}{C_t}\right)^{-\rho\frac{1-\alpha}{1-\rho}}\left(R_{t+1}^c\right)^{\frac{\rho-\alpha}{1-\rho}} \tag{46}$$

Instead, we rewrite the return on the wealth portfolio in terms of the wealth-consumption ratio WC

$$R_{t+1}^c = \frac{WC_{t+1}}{WC_t - 1}\frac{C_{t+1}}{C_t} = \frac{e^{wc_{t+1}}}{e^{wc_t} - 1}\frac{C_{t+1}}{C_t},$$

Substituting this into equation (46) delivers an expression that depends only on consumption growth and the wealth-consumption ratio:

$$M_{t+1} = \beta^{\frac{1-\alpha}{1-\rho}}\left(\frac{C_{t+1}}{C_t}\right)^{-\alpha}\left(\frac{WC_{t+1}}{WC_t - 1}\right)^{\frac{\rho-\alpha}{1-\rho}} = \beta^{\frac{1-\alpha}{1-\rho}}\left(\frac{C_{t+1}}{C_t}\right)^{-\alpha}\left(\frac{e^{wc_{t+1}}}{e^{wc_t} - 1}\right)^{\frac{\rho-\alpha}{1-\rho}} \tag{47}$$

□

The log SDF expression in the Bansal and Yaron (2004) model is a special case of this general, non-linear formulation. Indeed, one recovers the log SDF equation (48) below by using a first-order Taylor approximation of wc_t in equation (42) around A_0^c.

$$m_{t+1} = \frac{1-\alpha}{1-\rho}\left[\log\beta + \kappa_0^c\right] - \kappa_0^c - \alpha\Delta c_{t+1} - \frac{\alpha-\rho}{1-\rho}\left(wc_{t+1} - \kappa_1^c wc_t\right) \tag{48}$$

This expression follows immediately from the results in Appendix D.2 below. Since κ_1^c turns out to be essentially 1 under the Bansal and Yaron (2004) calibration, the second asset pricing factor in the SDF is essentially the log change in the wealth-consumption ratio. A second special case obtains by approximating the last term in (42) using a first-order Taylor expansion of wc_{t+1} around wc_t instead. In that case, we obtain a three-factor model:

$$m_{t+1} \approx \frac{1-\alpha}{1-\rho}\log\beta - \alpha\Delta c_{t+1} + \frac{\rho-\alpha}{1-\rho}\log\left(\frac{e^{wc_t}}{e^{wc_t}-1}\right) + \frac{\rho-\alpha}{1-\rho}\Delta wc_{t+1}. \tag{49}$$

Expressions (49) and (48) are functionally similar because κ_1^c is close to 1 and κ_0^c equals $\frac{e^{wc_t}}{e^{wc_t}-1}$ when wc_t is evaluated at its long-run mean A_0.

D.2 Proof of Proposition 3

Setting Up Notation The starting point of the analysis is the Euler equation $E_t[M_{t+1}R_{t+1}^i] = 1$, where R_{t+1}^i denotes a gross return between dates t and $t+1$ on some asset i and M_{t+1} is the SDF. In logs:

$$E_t\left[m_{t+1}\right] + E_t\left[r_{t+1}^i\right] + \frac{1}{2}Var_t\left[m_{t+1}\right] + \frac{1}{2}Var_t\left[r_{t+1}^i\right] + Cov_t\left[m_{t+1}, r_{t+1}^i\right] = 0. \tag{50}$$

The same equation holds for the real risk-free rate $y_t(1)$, so that

$$y_t(1) = -E_t\left[m_{t+1}\right] - \frac{1}{2}Var_t\left[m_{t+1}\right]. \tag{51}$$

The expected excess return becomes:

$$E_t\left[r_{t+1}^{e,i}\right] = E_t\left[r_{t+1}^i - y_t(1)\right] + \frac{1}{2}Var_t\left[r_{t+1}^i\right] = -Cov_t\left[m_{t+1}, r_{t+1}^i\right] = -Cov_t\left[m_{t+1}, r_{t+1}^{e,i}\right], \tag{52}$$

where $r_{t+1}^{e,i}$ denotes the excess return on asset i corrected for the Jensen term.

We adopt the consumption growth specification of Bansal and Yaron (2004), repeated from the main text:

$$\Delta c_{t+1} = \mu_c + x_t + \sigma_t\eta_{t+1}, \tag{53}$$

$$x_{t+1} = \rho_x x_t + \varphi_e\sigma_t e_{t+1}, \tag{54}$$

$$\sigma_{t+1}^2 = \bar{\sigma}^2 + \nu_1(\sigma_t^2 - \sigma^2) + \sigma_w w_{t+1}, \tag{55}$$

where (η_t, e_t, w_t) are i.i.d. mean-zero, variance-one, normally distributed innovations.

Proof of Linearity The proof closely follows the proof in Bansal and Yaron (2004), henceforth BY, but adjusts all expressions for our timing of returns.

Proof. In what follows we focus on the return on a claim to aggregate consumption, denoted r^c, where

$$r_{t+1}^c = \kappa_0^c + \Delta c_{t+1} + wc_{t+1} - \kappa_1^c wc_t,$$

and derive the five terms in equation (50) for this asset.

Taking logs on both sides of the non-linear SDF expression in equation (46) of Appendix D.1 delivers an expression of the log SDF as a function of log consumption changes and the log total wealth return

$$m_{t+1} = \frac{1-\alpha}{1-\rho}\log\beta - \frac{1-\alpha}{1-\rho}\rho\Delta c_{t+1} + \left(\frac{1-\alpha}{1-\rho} - 1\right) r^c_{t+1}. \tag{56}$$

We conjecture that the log wealth-consumption ratio is linear in the two states x_t and $\sigma_t^2 - \bar{\sigma}^2$,

$$wc_t = A_0^c + A_1^c x_t + A_2^c\left(\sigma_t^2 - \bar{\sigma}^2\right).$$

As BY, we assume joint conditional normality of consumption growth, x, and the variance of consumption growth. We verify this conjecture from the Euler equation (50).

Using the conjecture for the wealth-consumption ratio, we compute innovations in the total wealth return, and its conditional mean and variance:

$$\begin{aligned}
r^c_{t+1} - E_t\left[r^c_{t+1}\right] &= \sigma_t\eta_{t+1} + A_1^c\varphi_e\sigma_t e_{t+1} + A_2^c\sigma_w w_{t+1} \\
E_t\left[r^c_{t+1}\right] &= r_0 + \left(1 - (\kappa_1^c - \rho_x)A_1^c\right)x_t - A_2^c(\kappa_1^c - \nu_1)\left(\sigma_t^2 - \bar{\sigma}^2\right) \\
V_t\left[r^c_{t+1}\right] &= (1 + (A_1^c\varphi_e)^2)\sigma_t^2 + (A_2^c)^2\sigma_w^2 \\
r_0 &= \kappa_0^c + A_0^c(1-\kappa_1^c) + \mu_c
\end{aligned}$$

These equations correspond to (A.8) and (A.9) in the Appendix of BY.

Substituting in the expression for the log total wealth return r^c into the log SDF, we compute innovations, and the conditional mean and variance of the log SDF:

$$\begin{aligned}
m_{t+1} - E_t\left[m_{t+1}\right] &= -\alpha\sigma_t\eta_{t+1} - \frac{\alpha-\rho}{1-\rho}A_1^c\varphi_e\sigma_t e_{t+1} - \frac{\alpha-\rho}{1-\rho}A_2^c\sigma_w w_{t+1}, \\
E_t\left[m_{t+1}\right] &= m_0 - \rho x_t + \frac{\alpha-\rho}{1-\rho}(\kappa_1^c - \nu_1)A_2^c\left(\sigma_t^2 - \bar{\sigma}^2\right) \\
V_t\left[m_{t+1}\right] &= \left(\alpha^2 + \left(\frac{\alpha-\rho}{1-\rho}\right)^2 (A_1^c\varphi_e)^2\right)\sigma_t^2 + \left(\frac{\alpha-\rho}{1-\rho}\right)^2 (A_2^c)^2\sigma_w^2 \\
m_0 &= \frac{1-\alpha}{1-\rho}\log\beta - \frac{\alpha-\rho}{1-\rho}\left[\kappa_0^c + A_0^c(1-\kappa_1^c)\right] - \alpha\mu_c
\end{aligned} \tag{57}$$

These expressions correspond to equations (A.10) and (A.27), and are only slightly different due to the different timing in returns.

The conditional covariance between the log consumption return and the log SDF is given by the conditional expectation of the product of their innovations

$$Cov_t\left[r^c_{t+1}, m_{t+1}\right] = E_t\left[r^c_{t+1} - E_t\left[r^c_{t+1}\right], m_{t+1} - E_t\left[m_{t+1}\right]\right] = \left(-\alpha - \frac{\alpha-\rho}{1-\rho}(A_1^c\varphi_e)^2\right)\sigma_t^2 - \frac{\alpha-\rho}{1-\rho}(A_2^c)^2\sigma_w^2$$

Using the method of undetermined coefficients and the five components of equation (50), we can solve for the constants A_0^c, A_1^c, and A_2^c:

$$
\begin{aligned}
A_1^c &= \frac{1-\rho}{\kappa_1^c - \rho_x}, && (58)\\
A_2^c &= \frac{(1-\rho)(1-\alpha)}{2(\kappa_1^c - \nu_1)}\left[1 + \frac{\varphi_e^2}{(\kappa_1^c - \rho_x)^2}\right], && (59)\\
0 &= \frac{1-\alpha}{1-\rho}\left[\log\beta + \kappa_0^c + (1-\kappa_1^c)A_0^c\right] + (1-\alpha)\mu_c + \frac{1}{2}(1-\alpha)^2\left[1 + \frac{\varphi_e^2}{(\kappa_1^c - \rho_x)^2}\right]\bar{\sigma}^2 + \frac{1}{2}\left(\frac{1-\alpha}{1-\rho}\right)^2 (A_2^c)^2\sigma_w^2 && (60)
\end{aligned}
$$

The first two correspond to equations (A.5) and (A.7) in BY. The last equation implicitly solves A_0^c as a function of all parameters of the model. Because κ_0^c and κ_1^c are non-linear functions of A_0^c, this system of three equations needs to be solved simultaneously and numerically. Our computations indicate that the system has a unique solution. This verifies the conjecture that the log wealth-consumption ratio is linear in the two state variables. □

According to (52), the risk premium on the consumption claim is given by

$$
E_t\left[r_{t+1}^e\right] = E_t\left[r_{t+1}^c - y_t(1)\right] + .5V_t[r_{t+1}^c] = -\lambda_{m,\eta}\sigma_t^2 + \lambda_{m,e}B\sigma_t^2 + \lambda_{m,w}A_2^c\sigma_w^2, \quad (61)
$$

This corresponds to equation (A.11) in BY, with $\lambda_{m,\eta} = -\alpha$, $\lambda_{m,e} = \frac{\alpha-\rho}{1-\rho}A_1^c\varphi_e$, and $\lambda_{m,w} = \frac{\alpha-\rho}{1-\rho}A_2^c$. According to equation (51), the expression for the risk-free rate is given by

$$
\begin{aligned}
y_t(1) &= h_0 + \rho x_t + h_1(\sigma_t^2 - \bar{\sigma}^2) && (62)\\
h_0 &= -m_0 - .5\lambda_{m,w}^2\sigma_w^2 - .5\left(\lambda_{m,\eta}^2 + \lambda_{m,e}^2\right)\bar{\sigma}^2\\
h_1 &= -\frac{\alpha-\rho}{1-\rho}(\kappa_1^c - \nu_1)A_2^c - .5\left(\lambda_{m,\eta}^2 + \lambda_{m,e}^2\right)\\
&= .5(\rho-\alpha)\left(1 + \frac{\varphi_e^2}{(\kappa_1^c - \rho_x)^2}\right) - .5\left(\alpha^2 + (\alpha-\rho)^2\frac{\varphi_e^2}{(\kappa_1^c - \rho_x)^2}\right)
\end{aligned}
$$

This corresponds to equation (A.25-A.27) in BY. Its unconditional mean is simply h_0 (see A.28).

D.3 Quarterly Calibration LRR Model

The Bansal-Yaron model is calibrated and parameterized to monthly data. Since we want to use data on quarterly consumption and dividend growth, and a quarterly series for the wealth-consumption ratio, we recast the model at quarterly frequencies. We assume that the quarterly process for consumption growth, dividend growth, the low frequency component and the variance has the exact same structure than at the monthly frequency, with mean zero, standard deviation 1 innovations, but with different parameters. This appendix explains how the monthly parameters map into quarterly parameters. We denote all variables, shocks, and all parameters of the quarterly system with a tilde superscript. Our parameter values are listed at the end of this section, together with details on the simulation approach.

Preference Parameters Obviously, the preference parameters do not depend on the horizon ($\tilde{\alpha} = \alpha$ and $\tilde{\rho} = \rho$), except for the time discount factor $\tilde{\beta} = \beta^3$. Also, the long-run average log wealth-consumption ratio at the quarterly frequency is lower than at the monthly frequency by approximately log(3), because log of quarterly consumption is the log of three times monthly consumption.

Cash-flow Parameters We accomplish this by matching the conditional and unconditional mean and variance of log consumption and dividend growth. Log quarterly consumption growth is the sum of log consumption growth of three consecutive months. We obtain $\Delta\tilde{c}_{t+1} \equiv \Delta c_{t+3} + \Delta c_{t+2} + \Delta c_{t+1}$

$$\Delta\tilde{c}_{t+1} = 3\mu_c + (1+\rho_x+\rho_x^2)x_t + \sigma_t\eta_{t+1} + \sigma_{t+1}\eta_{t+2} + \sigma_{t+2}\eta_{t+3} + (1+\rho_x)\varphi_e\sigma_t e_{t+1} + \varphi_e\sigma_{t+1}e_{t+2} \tag{63}$$

Log quarterly dividend growth looks similar:

$$\Delta\tilde{d}_{t+1} = 3\mu_d + \phi(1+\rho_x+\rho_x^2)x_t + \varphi_d\sigma_t u_{t+1} + \varphi_d\sigma_{t+1}u_{t+2} + \varphi_d\sigma_{t+2}u_{t+3} + \phi(1+\rho_x)\varphi_e\sigma_t e_{t+1} + \phi\varphi_e\sigma_{t+1}e_{t+2} \tag{64}$$

First, we rescale the long-run component in the quarterly system, so that the coefficient on it in the consumption growth equation is still 1:

$$\tilde{x}_t = (1+\rho_x+\rho_x^2)x_t.$$

Second, we equate the unconditional mean of consumption and dividend growth :

$$\tilde{\mu} = 3\mu,\ \tilde{\mu}_d = 3\mu_d.$$

These imply that we also match the the conditional mean of consumption growth:

$$E_t[\Delta c_{t+3} + \Delta c_{t+2} + \Delta c_{t+1}] = 3\mu + (1+\rho_x+\rho_x^2)x_t = \tilde{\mu} + \tilde{x}_t = E_t[\Delta\tilde{c}_{t+1}]$$

Third, we also match the conditional mean of dividend growth by setting the quarterly leverage parameter $\tilde{\phi} = \phi$. Fourth, we match the unconditional variance of quarterly consumption growth:

$$\begin{aligned}
V[\Delta\tilde{c}_{t+1}] &= (1+\rho_x+\rho_x^2)^2 V[x_t] + \sigma^2\left[3 + (1+\rho_x)^2\varphi_e^2 + \varphi_e^2\right] \\
&= (1+\rho_x+\rho_x^2)^2 \frac{\varphi_e^2\sigma^2}{1-\rho_x^2} + \sigma^2\left[3 + (1+\rho_x)^2\varphi_e^2 + \varphi_e^2\right] \\
&= \frac{\tilde{\varphi}_e^2\tilde{\sigma}^2}{1-\tilde{\rho}_x^2} + \tilde{\sigma}^2
\end{aligned}$$

The first and second equalities use the law of iterated expectations to show that

$$V[\sigma_{t+j}\eta_{t+j+1}] \equiv E\left[E_{t+j}\left\{\sigma_{t+j}^2\eta_{t+j+1}^2\right\}\right] - \left(E\left[E_{t+j}\left\{\sigma_{t+j}\eta_{t+j+1}\right\}\right]\right)^2 = E\left[\sigma_{t+j}^2\right] - 0 = \sigma^2$$

and the same argument applies to terms of type $V[\sigma_{t+j}e_{t+j+1}]$. Coefficient matching on the variance of consumption expression delivers expressions for $\tilde{\sigma}^2$ and $\tilde{\varphi}_e$:

$$\tilde{\sigma}^2 = \sigma^2\left[3 + (1+\rho_x)^2\varphi_e^2 + \varphi_e^2\right]$$

$$\begin{aligned}
\tilde{\varphi}_e^2 &= \varphi_e^2 \frac{(1-\tilde{\rho}_x^2)(1+\rho_x+\rho_x^2)^2}{1-\rho_x^2}\frac{\sigma^2}{\tilde{\sigma}^2} \\
&= \frac{(1-\rho_x^6)(1+\rho_x+\rho_x^2)^2}{1-\rho_x^2}\frac{\varphi_e^2}{3 + (1+\rho_x)^2\varphi_e^2 + \varphi_e^2},
\end{aligned}$$

where the third equality uses the first equality. Note that we imposed $\tilde{\rho}_x = \rho_x^3$, which follows from a desire to match the persistence of the long-run cash-flow component. Recursively substituting, we find that the three-month ahead

x process has the following relationship to the current value:

$$x_{t+3} = \rho_x^3 x_t + \varphi_e \sigma_{t+2} e_{t+3} + \rho_x \varphi_e \sigma_{t+1} e_{t+2} + \rho_x^2 \varphi_e \sigma_t e_{t+1}$$

which compares to the quarterly equation

$$\tilde{x}_{t+1} = \tilde{\rho}_x \tilde{x}_t + \tilde{\varphi}_e \tilde{\sigma}_t \tilde{e}_{t+1}$$

The two processes now have the same auto-correlation and unconditional variance.

Fifth, we match the unconditional variance of dividend growth. Given the assumptions we have made sofar, this pins down φ_d:

$$\tilde{\varphi}_d^2 = \frac{3\varphi_d^2 + \phi^2(1+\rho_x)^2\varphi_e^2 + \phi^2\varphi_e^2}{3 + (1+\rho_x)^2\varphi_e^2 + \varphi_e^2}$$

Sixth, we match the autocorrelation and the unconditional variance of economic uncertainty σ_t^2. Iterating forward, we obtain an expression that relates variance in month t to the one in month $t+3$:

$$\sigma_{t+3}^2 - \sigma^2 = \nu_1^3(\sigma_t^2 - \sigma^2) + \sigma_w \nu_1^2 w_{t+1} + \sigma_w \nu_1 w_{t+2} + \sigma_w w_{t+3}$$

By setting $\tilde{\nu}_1 = \nu_1^3$ and $\tilde{\sigma}_w^2 = \sigma_w^2(1 + \nu_1^2 + \nu_1^4)$, we match the autocorrelation and variance of the quarterly equation

$$\tilde{\sigma}_{t+1}^2 - \tilde{\sigma}^2 = \tilde{\nu}_1(\tilde{\sigma}_t^2 - \tilde{\sigma}^2) + \tilde{\sigma}_w \tilde{w}_{t+1}$$

Parameter Values We use $\rho = 2/3$, $\alpha = 10$, and $\beta = .997$ for preferences; and $\mu_c = .45e^{-2}$, $\bar{\sigma} = 1.35e^{-2}$, $\rho_x = .938$, $\varphi_e = .126$, $\nu_1 = .962$, and $\sigma_w = .39 * 10^{-5}$ for the cash-flow processes in (14)-(16). The vector $\Theta^{LRR} = (\alpha, \rho, \beta, \mu_c, \bar{\sigma}, \varphi_e, \rho_x, \nu_1, \sigma_w)$ stores these parameters. The corresponding monthly values are $\Theta^{LRR} = \left(10, .6666, .998985, .0015, .0078, .044, .979, .987, .23 * 10^{-5}\right)$. A simulation of the quarterly model recovered the annualized cash-flow and asset return moments of the monthly simulation.

The quarterly calibration implies the following solution to the system of equations in (58)-(60): $A_0^{c,LRR} = 5.85$, $A_1^{c,LRR} = [5.16, -175.10]$. The corresponding linearization constants are $\kappa_0^c = .0198$ and $\kappa_1^c = 1.0029$.

The quarterly parameters of the dividend claim are: $\phi = 3$ and $\varphi_d = 4.4960$.

Simulation Most population moments of interest are known in closed-form in the LRR model, so that we do not have to simulate. However, the simulation approach has the advantages of generating small-sample biases that may also exist in the data and delivering (bootstrap) standard errors.

We run 5,000 simulations of the model for 236 quarters each, corresponding to the period 1948-2006. In each simulation we draw a 236×3 matrix of mutually uncorrelated standard normal random variables for the cash-flow innovations (η, e, w) in (14)-(16). We start off each run at the steady-state ($x_0 = 0$ and $\sigma_t^2 = \bar{\sigma}^2$). For each run, we form log consumption growth Δc_t, the two state variables $[x_t, \sigma_t^2 - \bar{\sigma}^2]$, the log wealth-consumption ratio wc_t and its first difference, and the log total wealth return r_t^c. We compute their first and second moments. These moments are based on the last 220 quarters only, for consistency with the length for our data for consumption growth and the growth rate of the wealth-consumption ratio (1952.I-2006.IV). This has the added benefit that the first 16 quarters are "burn-in," so that the first observation we use for the state vector is different in each run.

D.4 Campbell-Shiller Decomposition

Expected discounted future returns and consumption growth rates are given by:

$$
\begin{aligned}
r_t^H &\equiv E_t\left[\sum_{j=1}^{\infty}(\kappa_1^c)^{-j}\, r_{t+j}\right] = \frac{r_0^c}{\kappa_1^c-1} + \frac{\rho}{\kappa_1^c-\rho_x}x_t - A_2^c(\sigma_t^2-\bar{\sigma}^2) \\
\Delta c_t^H &\equiv E_t\left[\sum_{j=1}^{\infty}(\kappa_1^c)^{-j}\, \Delta c_{t+j}\right] = \frac{\mu_c}{\kappa_1^c-1} + \frac{1}{\kappa_1^c-\rho_x}x_t
\end{aligned}
$$

These expressions use the definitions of the total wealth return and consumption, as well as their dynamics. Starting from the affine expression for wc, we can write the wc ratio as a constant plus expected future consumption growth minus expected future total wealth returns:

$$
\begin{aligned}
wc_t &= A_0^c + A_1^c x_t + A_2^c(\sigma_t^2-\bar{\sigma}^2) = A_0^c + \frac{1}{\kappa_1^c-\rho_x}x_t - \left(\frac{\rho}{\kappa_1^c-\rho_x}x_t - A_2^c(\sigma_t^2-\bar{\sigma}^2)\right) \\
&= A_0^c + \left(\Delta c_t^H - \frac{\mu_c}{\kappa_1^c-1}\right) - \left(r_t^H - \frac{r_0^c}{\kappa_1^c-1}\right) = \frac{\kappa_0^c}{\kappa_1^c-1} + \Delta c_t^H - r_t^H
\end{aligned}
$$

The second equality uses the definition of A_1^c. The third equality uses the definition of r_t^H and Δc_t^H. The fourth equality uses the definition of r_0^c.

The variance of the log wealth-consumption ratio can be written in two equivalent ways:

$$
V\left[\Delta c_t^H\right] + V\left[r_t^H\right] - 2Cov\left[r_t^H, \Delta c_t^H\right] = V\left[wc_t\right] = Cov\left[wc_t, \Delta c_t^H\right] + Cov\left[wc_t, -r_t^H\right]
$$

In the LRR model, the terms in this expression are given by

$$
\begin{aligned}
V\left[\Delta c_t^H\right] &= \frac{1}{(\kappa_1^c-\rho_x)^2}\frac{\varphi_e^2}{1-\rho_x^2}\bar{\sigma}^2 > 0 \\
V\left[r_t^H\right] &= \frac{\rho^2}{(\kappa_1^c-\rho_x)^2}\frac{\varphi_e^2}{1-\rho_x^2}\bar{\sigma}^2 + (A_2^c)^2\frac{\sigma_w^2}{1-\nu_1^2} > 0 \\
Cov\left[r_t^H, \Delta c_t^H\right] &= \frac{\rho}{(\kappa_1^c-\rho_x)^2}\frac{\varphi_e^2}{1-\rho_x^2}\bar{\sigma}^2 > 0 \\
Cov\left[wc_t, \Delta c_t^H\right] &= \frac{1-\rho}{(\kappa_1^c-\rho_x)^2}\frac{\varphi_e^2}{1-\rho_x^2}\bar{\sigma}^2 > 0 \quad \Leftrightarrow \rho < 1 \\
Cov\left[wc_t, -r_t^H\right] &= \frac{\rho^2-\rho}{(\kappa_1^c-\rho_x)^2}\frac{\varphi_e^2}{1-\rho_x^2}\bar{\sigma}^2 + (A_2^c)^2\frac{\sigma_w^2}{1-\nu_1^2} > 0 \quad \Leftarrow \rho > 1
\end{aligned}
$$

We can break up expected future returns into a risk-free rate component and a risk premium component. The former is equal to

$$
E_t\left[\sum_{j=1}^{\infty}(\kappa_1^c)^{-j}\, r_{t+j-1}^f\right] = \frac{h_0}{\kappa_1^c-1} + \frac{\rho}{\kappa_1^c-\rho_x}x_t + \frac{h_1}{\kappa_1^c-\nu_1}(\sigma_t^2-\bar{\sigma}^2), \tag{65}
$$

where the second equation uses the expression for the risk-free rate in equation (62) to compute future risk-free rates and takes their time-t expectations. The risk premium component is simply the difference between r_t^H and the second expression.

D.5 Pricing Stocks in the LRR Model

Dividend Growth Process We start by pricing a claim to aggregate dividends, where the dividend process follows the specification in Bansal and Yaron (2004):

$$\Delta d_{t+1} = \mu_d + \phi x_t + \varphi_d \sigma_t u_{t+1} \tag{66}$$

The shock u_t is orthogonal to (η, e, w).

Defining returns ex-dividend and using the Campbell (1991) linearization, the log return on a claim to the aggregate dividend can be written as:

$$r^m_{t+1} = \Delta d_{t+1} + pd^m_{t+1} + \kappa^m_0 - \kappa^m_1 pd^m_t,$$

with coefficients

$$\kappa^m_1 = \frac{e^{A^m_0}}{e^{A^m_0} - 1} > 1, \quad \text{and} \quad \kappa^m_0 = -\log\left(e^{A^m_0} - 1\right) + \frac{e^{A^m_0}}{e^{A^m_0} - 1} A^m_0$$

which depend on the long-run log price-dividend ratio A^m_0.

Proof of Linearity We conjecture, as we did for the wealth-consumption ratio, that the log price dividend ratio is linear in the two state variables:

$$pd^m_t = A^m_0 + A^m_1 x_t + A^m_2 (\sigma^2_t - \bar{\sigma}^2).$$

As we did for the return on the consumption claim, we compute innovations in the dividend claim return, and its conditional mean and variance:

$$\begin{aligned}
r^m_{t+1} - E_t\left[r^m_{t+1}\right] &= \varphi_d \sigma_t u_{t+1} + \beta_{m,e}\sigma_t e_{t+1} + \beta_{m,w}\sigma_w w_{t+1} \\
E_t\left[r^m_{t+1}\right] &= r^m_0 + [\phi + A^m_1(\rho_x - \kappa^m_1)]x_t - A^m_2(\kappa^m_1 - \nu_1)\left(\sigma^2_t - \bar{\sigma}^2\right) \\
V_t\left[r^m_{t+1}\right] &= (\varphi^2_d + \beta^2_{m,e})\sigma^2_t + \beta^2_{m,w}\sigma^2_w, \\
r^m_0 &= \kappa^m_0 + A^m_0(1 - \kappa^m_1) + \mu_d
\end{aligned}$$

where $\beta_{m,e} = A^m_1 \varphi_e$ and $\beta_{m,w} = A^m_2$. These equations correspond to (A.12) and (A.13) in the Appendix of Bansal and Yaron (2004). Finally, the conditional covariance between the log SDF and the log dividend claim return is

$$Cov_t\left[m_{t+1}, r^m_{t+1}\right] = -\lambda_{m,e}\beta_{m,e}\sigma^2_t - \lambda_{m,w}\beta_{m,w}\sigma^2_w$$

From the Euler equation for this return $E_t[m_{t+1}] + E_t\left[r^m_{t+1}\right] + \frac{1}{2}V_t[m_{t+1}] + \frac{1}{2}V_t\left[r^m_{t+1}\right] + Cov_t[m_{t+1}, r^m_{t+1}] = 0$ and the method of undetermined coefficients, we can use the same procedure as described in D.2, and solve for the constants A^m_0, A^m_1, and A^m_2:

$$\begin{aligned}
A^m_1 &= \frac{\phi - \rho}{\kappa^m_1 - \rho_x}, \\
A^m_2 &= \frac{\left[\frac{\alpha - \rho}{1 - \rho} A^c_2(\kappa_1 - \nu_1) + .5 H_m\right]}{\kappa^m_1 - \nu_1}, \\
0 &= m_0 + \kappa^m_0 + (1 - \kappa^m_1)A^m_0 + \mu_d + \frac{1}{2}H_m\bar{\sigma}^2 + \frac{1}{2}\left(A^m_2 - A^c_2\frac{\alpha - \rho}{1 - \rho}\right)^2 \sigma^2_w
\end{aligned}$$

where

$$H_m = \alpha^2 + \left(A_1^m \varphi_e + (\frac{\rho-\alpha}{1-\rho}) A_1^c \varphi_e\right)^2 + \varphi_d^2$$

Again, this is a non-linear system in three equations and three unknowns, which we solve numerically. The first two equations correspond to (A.16) and (A.20) in BY.

Equity Risk premium and CS Decomposition The equity risk premium on the dividend claim (adjusted for a Jensen term) becomes:

$$E_t\left[r_{t+1}^{e,m}\right] \equiv E_t\left[r_{t+1}^m - y_t(1)\right] + .5V_t[r_{t+1}^m] = \lambda_{m,e}\beta_{m,e}\sigma_t^2 + \lambda_{m,w}\beta_{m,w}\sigma_w^2 \tag{67}$$

This corresponds to equation (A.14) in BY.

Expected discounted future equity returns and dividend growth rates are given by:

$$r_t^{m,H} \equiv E_t\left[\sum_{j=1}^{\infty} (\kappa_1^m)^{-j} r_{t+j}^m\right] = \frac{r_0^m}{\kappa_1^m - 1} + \frac{\rho}{\kappa_1^m - \rho_x} x_t - A_2^m(\sigma_t^2 - \bar{\sigma}^2) \tag{68}$$

$$\Delta d_t^H \equiv E_t\left[\sum_{j=1}^{\infty} (\kappa_1^m)^{-j} \Delta d_{t+j}\right] = \frac{\mu_d}{\kappa_1^m - 1} + \frac{\phi}{\kappa_1^m - \rho_x} x_t \tag{69}$$

From these expressions, it is easy to see that $pd_t^m = \frac{\kappa_0^m}{\kappa_1^m - 1} + \Delta d_t^H - r_t^{m,H}$, and to compute the elements of the variance-decomposition:

$$V[pd_t^m] = Cov[pd_t^m, \Delta d_t^H] + Cov[pd_t^m, -r_t^{m,H}] = V[\Delta d_t^H] + V[r_t^{m,H}] - 2Cov[\Delta d_t^H, r_t^{m,H}].$$

E The External Habit Model

The organization of this EH model appendix exactly parallels the treatment of the LRR model in Appendix (D.2).

E.1 Proof of Proposition 4

Proof. We conjecture that the log wealth-consumption ratio is linear in the sole state variable $(s_t - \bar{s})$,

$$wc_t = A_0^c + A_1^c (s_t - \bar{s}).$$

As Campbell and Cochrane (1999), henceforth CC, we assume joint conditional normality of consumption growth and the surplus consumption ratio. We verify this conjecture from the Euler equation for total wealth.

We start from the canonical log SDF in the external habit model:

$$m_{t+1} = \log\beta - \alpha\Delta c_{t+1} - \alpha\Delta s_{t+1}.$$

Substituting in the expression for returns into the log SDF, we compute innovations, and the conditional mean and variance of the log SDF:

$$\begin{aligned} m_{t+1} - E_t\left[m_{t+1}\right] &= -\alpha(1+\lambda_t)\bar{\sigma}\eta_{t+1} \\ E_t\left[m_{t+1}\right] &= m_0 + \alpha(1-\rho_s)\left(s_t - \bar{s}\right) \\ V_t\left[m_{t+1}\right] &= \alpha^2\left(1+\lambda_t\right)^2\bar{\sigma}^2 \\ m_0 &= \log\beta - \alpha\mu_c \end{aligned} \tag{70}$$

Likewise, we compute innovations in the consumption claim return, and its conditional mean and variance:

$$\begin{aligned} r^c_{t+1} - E_t\left[r^c_{t+1}\right] &= (1+A^c_1\lambda_t)\bar{\sigma}\eta_{t+1} \\ E_t\left[r^c_{t+1}\right] &= r_0 - A^c_1(\kappa^c_1-\rho_s)\left(s_t-\bar{s}\right) \\ V_t\left[r^c_{t+1}\right] &= (1+A^c_1\lambda_t)^2\bar{\sigma}^2 \\ r_0 &= \kappa^c_0 + A^c_0(1-\kappa^c_1) + \mu_c \end{aligned} \tag{71}$$

The conditional covariance between the log consumption return and the log SDF is given by the conditional expectation of the product of their innovations

$$Cov_t\left[m_{t+1}, r^c_{t+1}\right] = -\alpha\left(1+\lambda_t\right)\left(1+A^c_1\lambda_t\right)\bar{\sigma}^2$$

We assume that the sensitivity function takes the following form

$$\lambda_t = \frac{\bar{S}^{-1}\sqrt{1-2(s_t-\bar{s})}+1-\alpha}{\alpha - A^c_1}$$

Using the method of undetermined coefficients and the five components of equation (50), we can solve for the constants A^c_0 and A^c_1:

$$A^c_1 = \frac{(1-\rho_s)\alpha - \bar{\sigma}^2\bar{S}^{-2}}{\kappa^c_1 - \rho_s}, \tag{72}$$

$$0 = \log\beta + \kappa^c_0 + (1-\kappa^c_1)A^c_0 + (1-\alpha)\mu_c + .5\bar{\sigma}^2\bar{S}^{-2} \tag{73}$$

This verifies that our conjecture was correct. □

It follows immediately that the log SDF can be written as a function of consumption growth and the change in the log wealth-consumption ratio

$$m_{t+1} = \log\beta - \alpha\Delta c_{t+1} - \frac{\alpha}{A^c_1}\left(wc_{t+1} - wc_t\right).$$

The risk premium on the consumption claim is given by $Cov_t[r^c_{t+1}, -m_{t+1}]$:

$$E_t\left[r^e_{t+1}\right] \equiv E_t\left[r^c_{t+1} - y_t(1)\right] + .5V_t[r^c_{t+1}] = \alpha\left(1+\lambda_t\right)\left(1+A^c_1\lambda_t\right)\bar{\sigma}^2, \tag{74}$$

where the second term on the left is a Jensen adjustment. The expression for the risk-free rate appears in the next section E.2.

E.2 The Steady-State Habit Level

Campbell and Cochrane (1999) engineer their sensitivity function λ_t to deliver a risk-free rate that is linear in the state $s_t - \bar{s}$. (They mostly study a special case with a constant risk-free rate.) The linearity of the risk-free rate is accomplished by choosing

$$\lambda_t^{CC} = \bar{S}^{-1}\sqrt{1-2(s_t-\bar{s})} - 1 \tag{75}$$

Note that if the risk aversion parameter $\alpha = 2$ and $A_1^c = 1$, our sensitivity function exactly coincides with CC's. Instead, we engineer our sensitivity function to deliver a log wealth-consumption ratio that is linear in $s_t - \bar{s}$. As a result of our choice, the risk-free rate, $y_t(1) = -E_t[m_{t+1}] - .5V_t[m_{t+1}]$, is no longer linear in the state, but contains an additional square-root term:

$$\begin{aligned} y_t(1) &= h_0 + \left[\frac{\bar{\sigma}^2\alpha^2\bar{S}^{-2}}{(\alpha - A_1^c)^2} - \alpha(1-\rho_s)\right](s_t - \bar{s}) - \bar{\sigma}^2\alpha^2\frac{(1-A_1^c)\bar{S}^{-1}}{(\alpha-A_1^c)^2}\left(\sqrt{1-2(s_t-\bar{s})} - 1\right) && (76) \\ h_0 &= -\log\beta + \alpha\mu_c - .5\bar{\sigma}^2\alpha^2(1+\lambda(\bar{s}))^2, \quad \text{where} \quad \lambda(\bar{s}) = \left(\frac{\bar{S}^{-1} + 1 - \alpha}{\alpha - A_1^c}\right) && (77) \end{aligned}$$

where $\lambda(\bar{s})$ is obtained from evaluating our sensitivity function at $s_t = \bar{s}$. CC obtain a similar expression, but without the last term. If $\alpha = 2$ and $A_1^c = 1$, the expression collapses to the one in CC. A constant risk-free rate obtains in the CC model when $\bar{S}^{-1} = \bar{\sigma}^{-1}\sqrt{\frac{1-\rho_s}{\alpha}}$ because this choice makes the linear term vanish. While there is no $\bar{S}$ that guarantees a constant risk-less interest rate under our assumptions, we choose $\bar{S}$ to match the *steady-state* risk-free rate in CC, $\bar{r} = -\log\beta + \alpha\mu - .5\alpha(1-\rho_s)$. That is, we set $s_t = \bar{s}$ in the above equation, which then collapses to h_0. Setting $\bar{r} = h_0$ allows us to solve for $\bar{S}^{-1}$ as a function of A_1^c and the structural parameters α, ρ_s, and $\bar{\sigma}$:

$$\bar{S}^{-1} = (\alpha - A_1^c)\left(\bar{\sigma}^{-1}\sqrt{\frac{1-\rho_s}{\alpha}}\right) - 1 + A_1^c. \tag{78}$$

Substituting this expression back into the sensitivity function (20), we find that the steady-state sensitivity level $\lambda(\bar{s}) = \bar{\sigma}^{-1}\sqrt{\frac{1-\rho_s}{\alpha}} - 1$. This implies that we generate the same steady-state conditional covariance between the surplus consumption ratio and consumption growth as in CC. As in CC, we define a maximum value for the log surplus consumption ratio s_{max}, as the value at which λ_t runs into zero:

$$s_{max} = \bar{s} + \frac{1}{2}\left(1 - (\alpha-1)^2\bar{S}^2\right)$$

Note that if $\alpha = 2$, this coincides with equation (11) in CC. It is understood that $\lambda_t = 0$ for $s_t \geq s_{max}$.

E.3 Alternative Way of Pinning Down $\bar{S}$

To conclude the discussion of the EH model, we investigate an alternative way to pin down $\bar{S}$. In our benchmark method, outlined in Appendix E.2, we chose it to match the steady state risk-free rate in Campbell and Cochrane (1999). Here, the alternative is to pin down $\bar{S}$ to match the average wealth-consumption ratio of 26.75 in Campbell and Cochrane (1999).

As before, we solve a system of three equations in $(A_0^c, A_1^c, \bar{S})$, only the third of which is different and simply imposes that $e^{A_0^c - \log(4)} = 26.75$. We obtain the following solution: $A_0^c = 4.673$, $A_1^c = 0.447$, and $\bar{S} = 0.0339$. The wealth-consumption ratio is higher and less sensitive to the surplus-consumption ratio than in the benchmark case. The volatility of the surplus-consumption ratio is 41.6%, similar to the benchmark model. Because A_1^c is lower, so is the volatility of the wc ratio. It is 18.6% in the model, still much higher than the 8.4% in the data. The volatilities of the change in the wc ratio and of the total wealth return are also lowered, but remain too high. Since, we are no longer pinning $\bar{S}$ down to match the steady-state risk-free rate, the risk-free rate turns negative: -33 basis points per quarter or -1.2% per year. It is also more volatile: .59% versus .03 in the main text and .55 in the data. The consumption risk premium is down from 2.67% per quarter to 1.97% per quarter and the equity premium is down from 3.30% per quarter to 2.23%. The main cost of this calibration is a price-dividend ratio that is too low. The volatility of pd^m is now only 12.5% per quarter compared to 27% in the data.

E.4 Quarterly Calibration EH Model

Preference Parameters Again, the preference parameter does not depend on the horizon ($\tilde{\alpha} = \alpha$, except for the time discount factor $\tilde{\beta} = \beta^3$). The surplus consumption ratio has the same law of motion as in the monthly model, but we set its persistence equal to $\tilde{\rho}_s = \rho_s^3$.

Cash-flow Parameters Following a similar logic, we can match mean and variance of quarterly consumption and dividend growth in the CC model. From matching the means we get:

$$\tilde{\mu} = 3\mu, \ \tilde{\mu}_d = 3\mu_d.$$

From matching the variances we get

$$\tilde{\sigma}^2 = 3\sigma^2, \ \tilde{\varphi}_d = \varphi_d, \ \tilde{\chi} = \chi.$$

Calibration We work with a quarterly calibration and set $\alpha = 2$, $\rho_s = .9658$, and $\beta = .971$ for preferences, and $\mu_c = .47e^{-2}$ and $\bar{\sigma} = .75e^{-2}$ for the cash-flow process (18). We summarize the parameters in the vector $\Theta^{EH} = (\alpha, \rho_s, \beta, \mu_c, \bar{\sigma})$. The corresponding monthly values are $\Theta^{EH} = (\alpha, \rho_s, \beta, \mu_c, \bar{\sigma}) = \left(2, .9885, .990336, .1575e^{-2}, .433e^{-2}\right)$.

We solve for the loadings of the state variables in the log wealth-consumption ratio form equations (72), (73), and (78), and find: $A_0^{EH} = 3.86$, $A_1^{EH} = 0.778$, and $\bar{S} = .0474$. The corresponding Campbell-Shiller linearization constants are $\kappa_0^c = .1046$ and $\kappa_1^c = 1.021583$.

For the dividend process described below, we set $\mu_d = \mu$, $\varphi_d = 7.32$, and $\chi = 0.20$.

A simulation of the quarterly model recovered the annualized cash-flow and asset return moments of the monthly simulation.

E.5 Campbell-Shiller Decomposition

Using (71) and the law of motion for s_t and consumption growth, expected discounted future returns and consumption growth rates are given by:

$$\begin{aligned} r_t^H &\equiv E_t\left[\sum_{j=1}^{\infty} (\kappa_1^c)^{-j}\, r_{t+j}\right] = \frac{r_0^c}{\kappa_1^c - 1} - A_1^c(s_t - \bar{s}) \\ \Delta c_t^H &\equiv E_t\left[\sum_{j=1}^{\infty} (\kappa_1^c)^{-j}\, \Delta c_{t+j}\right] = \frac{\mu_c}{\kappa_1^c - 1} \end{aligned}$$

These expressions enable us to go back and forth between the affine log wealth-consumption ratio expression and the Campbell-Shiller decomposition:

$$\begin{aligned} wc_t &= A_0^c + A_1^c(s_t - \bar{s}) = A_0^c + \left(\Delta c_t^H - \frac{\mu_c}{\kappa_1^c - 1}\right) - \left(r_t^H - \frac{r_0^c}{\kappa_1^c - 1}\right) \\ &= \frac{\kappa_0^c}{\kappa_1^c - 1} + \Delta c_t^H - r_t^H, \end{aligned}$$

The variance of the log wealth-consumption ratio can be written in two equivalent ways:

$$V\left[\Delta c_t^H\right] + V\left[r_t^H\right] - 2Cov\left[r_t^H, \Delta c_t^H\right] = V\left[wc_t\right] = Cov\left[wc_t, \Delta c_t^H\right] + Cov\left[wc_t, -r_t^H\right]$$

In the EH model, the terms in this expression are given by

$$\begin{aligned} V\left[\Delta c_t^H\right] &= 0, \;\; Cov\left[r_t^H, \Delta c_t^H\right] = 0, \;\; Cov\left[wc_t, \Delta c_t^H\right] = 0 \\ V\left[r_t^H\right] &= (A_1^c)^2 \left(\frac{\bar{S}^{-1} + 1 - \alpha}{\alpha - A_1^c}\right)^2 \frac{1}{1 - \rho_s^2} \bar{\sigma}^2 > 0 \\ Cov\left[wc_t, -r_t^H\right] &= (A_1^c)^2 \left(\frac{\bar{S}^{-1} + 1 - \alpha}{\alpha - A_1^c}\right)^2 \frac{1}{1 - \rho_s^2} \bar{\sigma}^2 > 0 \end{aligned}$$

Likewise, there is no predictability in dividend growth (see equation 79). Therefore, $V\left[pd_t\right] = V\left[r_t^{H,m}\right]$, where the latter is the unconditional variance of the expected return on the dividend claim.

E.6 Pricing Stocks in EH Model

The main difference with the analysis in the long-run risk model, and the analysis for the total wealth return in the EH model is that the return to the aggregate dividend claim *cannot* be written as a linear function of the state variables. Our choice of the sensitivity function makes the log wealth-consumption ratio linear in the surplus consumption ratio. But, for that same sensitivity function, the log price-dividend ratio is not linear in the surplus-consumption ratio. As a result, we need to resort to a non-linear computation of the price-dividend ratio on the aggregate dividend claim.

Dividend Growth Process In Campbell and Cochrane (1999), dividend growth is i.i.d., with the same mean μ as consumption growth, and innovations that are correlated with the innovations in consumption growth. To make the dividend growth process more directly comparable across models, we write it as a function of innovations to consumption growth η and innovations u that are orthogonal to η:

$$\Delta d_{t+1} = \mu_d + \varphi_d \bar{\sigma} u_{t+1} + \varphi_d \bar{\sigma} \chi \eta_{t+1}. \tag{79}$$

It follows immediately that its (un)conditional variance equals $\varphi_d^2 \bar{\sigma}^2 (1 + \chi^2)$ and its (un)conditional covariance with consumption growth is $\varphi_d \bar{\sigma}^2 \chi$. If correlation between consumption and dividend growth is $corr$, then $\chi = \sqrt{corr^2/(1 - corr^2)}$. We set φ_d and χ to replicate the unconditional variance of dividend growth and the correlation of dividend growth and consumption growth $corr$ in Campbell and Cochrane (1999).

Computation of Price-Dividend Ratio Wachter (2005) shows that the price-dividend ratio on a claim to aggregate dividends can be written as the sum of the price-dividend ratios on strips to the period-n dividend, for $n = 1, \cdots, \infty$:

$$\frac{P_t}{D_t} = \sum_{n=1}^{\infty} \frac{P^d_{nt}}{D_t} \tag{80}$$

We adopt her methodology and show it continues to hold for our slightly different dividend growth process in equation (79). A similar approach works when there is cointegration between consumption and dividends (Appendix A in Wachter (2005)).

The Euler equation for the period-n strip delivers the following expression for the price-dividend ratio

$$\frac{P^d_{nt}}{D_t} = E_t \left[M_{t+1} \frac{P^d_{n-1,t+1}}{D_{t+1}} \frac{D_{t+1}}{D_t} \right]$$

We conjecture that the price-dividend ratio on the period-n strip equals a function $F_n(s_t)$, which follows the recursion

$$F_n(s_t) = \beta e^{\mu_d - \alpha\mu_c + \alpha(1-\rho_s)(s_t - \bar{s}) + \frac{1}{2}\varphi_d^2 \bar{\sigma}^2} E_t \left[e^{[\varphi_d \chi - \alpha(1+\lambda_t)]\bar{\sigma}\eta_{t+1}} F_{n-1}(s_{t+1}) \right],$$

starting at $F_0(s_t) = 1$. We now verify this conjecture.

Proof. Substituting in the conjecture $\frac{P^d_{nt}}{D_t} = F_n(s_t)$ into the Euler equation for the period-n strip, we get

$$F_n(s_t) = E_t \left[M_{t+1} F_{n-1}(s_{t+1}) \frac{D_{t+1}}{D_t} \right].$$

Substituting in for the stochastic discount factor M and the dividend growth process (79), this becomes

$$F_n(s_t) = \beta e^{\mu_d - \alpha\mu_c + \alpha(1-\rho_s)(s_t - \bar{s})} E_t \left[e^{-\alpha(1+\lambda_t)\bar{\sigma}\eta_{t+1}} F_{n-1}(s_{t+1}) e^{\varphi_d \bar{\sigma} u_{t+1} + \varphi_d \chi \bar{\sigma} \eta_{t+1}} \right].$$

Because u and η are independent, we can write the expectation as a product of expectations. Because u is standard normal, the expectation in the previous expression can be written as

$$e^{\frac{1}{2}\varphi_d^2 \bar{\sigma}^2} E_t \left[e^{[\varphi_d \chi - \alpha(1+\lambda_t)]\bar{\sigma}\eta_{t+1}} F_{n-1}(s_{t+1}) \right].$$

This then verifies the conjecture for $F_n(s_t)$. $\square$

Finally, let $g(\eta)$ be the standard normal pdf, then we can compute this function through numerical integration

$$F_n(s_t) = \beta e^{\mu_d - \alpha\mu_c + \alpha(1-\rho_s)(s_t - \bar{s}) + \frac{1}{2}\varphi_d^2 \bar{\sigma}^2} \int_{-\infty}^{+\infty} e^{[\varphi_d \chi - \alpha(1+\lambda(s_t))]\bar{\sigma}\eta_{t+1}} F_{n-1}(s_{t+1}) g(\eta_{t+1}) d\eta_{t+1},$$

starting at $F_0(s_t) = 1$. The grid for s_t includes 14 very low values for s_t (-300, -250, -200, -150, -100, -50, -40, -30, -20, -15, -10, -9, -8, -7), 100 linearly spaced points between -6.5 and $\bar{s} * 1.001 = -2.85$, and the log of 100 linearly spaced points between $\bar{S}$ and $\exp(1.0000001 s_{max})$. The function evaluation $F_{n-1}(s_{t+1})$ is done using linear interpolation (and extrapolation) on the grid for the log surplus-consumption ratio s. The integral is computed in matlab using quad.m. The price dividend ratio is computed as the sum of the price-dividend ratios for the first 500 strips.