Appendix

To Accompany J. Andreoni and A. Brownback
"Grading on a Curve, and other Effects of Group Size on All-Pay Auctions"
For online publication only.

1 Proofs

Proposition 1: *Optimal Bidding Function is Weakly Monotonic in Valuation.*
Proof: Suppose not. Then v_i and v_j exist such that: $v_i < v_j$ but $b_i > b_j$.

Incentive compatibility for i dictates that

$$U(b_i, v_i, N) \geq U(b_j, v_i, N).$$

Substituting in the bidder's utility yields

$$v_i * P_N(b_i) - b_i \geq v_i * P_N(b_j) - b_j,$$

and rearranging we find

$$b_i - b_j \leq v_i \left(P_N(b_i) - P_N(b_j)\right).$$

A similar derivation for j reveals

$$b_i - b_j \geq v_j \left(P_N(b_i) - P_N(b_j)\right).$$

Therefore, since $b_i > b_j$, by construction this implies $P_N(b_i) \geq P_N(b_j)$. Subsequently, we have

$$v_i \left(P_N(b_i) - P_N(b_j)\right) \geq v_j \left(P_N(b_i) - P_N(b_j)\right).$$

Thus, $v_i \geq v_j$ or $P_N(b_i) = P_N(b_j)$. In the case of the former we have a contradiction. In the case of the latter, our contradiction is satisfied by a violation of incentive compatibility. Hence, $v_i < v_j \Longrightarrow b_i \leq b_j$. QED.

Corollary 1: *Monotonicity holds under Risk Aversion*

Proof: Again we start with:

$$U(b_i, v_i, N) \geq U(b_j, v_i, N)$$

and

$$U(b_j, v_j, N) \geq U(b_i, v_j, N).$$

Expand to

$$U(v_i - b_i) * P_N(b_i) + U(-b_i)\left(1 - P_N(b_i)\right) \geq U(v_i - b_j) * P_N(b_j) + U(-b_j)\left(1 - P_N(b_j)\right)$$

and

$$U(v_j - b_j) * P_N(b_j) + U(-b_j)\left(1 - P_N(b_j)\right) \geq U(v_j - b_i) * P_N(b_i) + U(-b_i)\left(1 - P_N(b_i)\right).$$

Solving for common terms and combining yields

$$\begin{aligned} U(v_i - b_i) * P_N(b_i) + U(-b_i)\left(1 - P_N(b_i)\right) - U(v_i - b_j) * P_N(b_j) \geq \\ U(v_j - b_i) * P_N(b_i) + U(-b_i)\left(1 - P_N(b_i)\right) - U(v_j - b_j) * P_N(b_j). \end{aligned}$$

Canceling terms and grouping simplifies this to

$$P_N(b_i)\left[U(v_i - b_i) - U(v_j - b_i)\right] \geq P_N(b_j)\left[U(v_i - b_j) - U(v_j - b_j)\right].$$

We have assumed $v_j > v_i$ and $b_i > b_j$, so we can define $v_i - b_i \equiv A$, $v_j - b_i \equiv A'$, $v_i - b_j \equiv B$, and $v_j - b_j \equiv B'$ with $A' > A$, $B' > B$, and $B > A$:

$$P_N(b_i)\left[U(A) - U(A')\right] \geq P_N(b_j)\left[U(B) - U(B')\right]$$

But, since $A' - A = B' - B$ and $U" < 0$ we know

$$\left[U(A) - U(A')\right] < \left[U(B) - U(B')\right] < 0.$$

Our assumptions also tell us that $P_N(b_j) < P_N(b_i)$,meaning

$$P_N(b_i)\left[U(A) - U(A')\right] < P_N(b_j)\left[U(B) - U(B')\right],$$

and we arrive at a contradiction. QED.

2 Alternative Assumptions on Utility

Here we reconsider our model under assumptions of risk averse bidders, and bidders with a joy-of-winning.

2.1 Risk Averse Bidding Functions

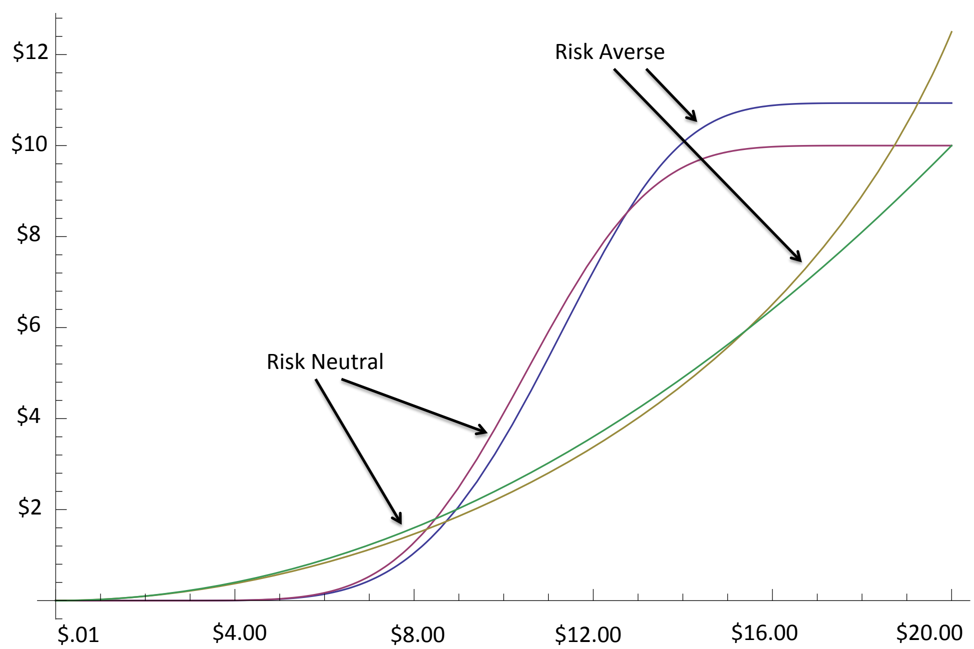

Figure 1: Risk Averse Bidding Equilibrium

This graph details the bidding function under CRRA utility of $U(x) = \frac{x^{1-\sigma}}{1-\sigma}$ with $\sigma = 3$. Here we can see that the risk aversion does not strongly affect the shape or magnitude of the bidding function in a symmetric equilibrium. Moreover, the risk averse Nash Equilibrium deviates from the data even more than does the risk neutral Nash Equilibrium over middle valuations.

2.2 Joy of Winning

We can modify our optimal bidding framework to include a constant utility gain simply from winning an auction. This will result in the following utility function:

$$U(\bullet) = (v_i + c) * P_N(b_i) - b_i$$

Resulting in the following bidding functions:

$$\begin{aligned}
B(v_i, 2) = &\frac{v_i^2}{2} + cv_i \\
B(v_i, 20) = &-(\frac{1}{2})x^{10}\big\{2c(48620x^9 - 461890x^8 + 1956240x^7 - 4849845x^6 + 7759752x^5 \\
&- 8314020x^4 + 5969040x^3 - 2771340x^2 + 755820x - 92378) \\
&+ x(92378x^9 - 875160x^8 + 3695120x^7 - 9129120x^6 + 14549535x^5 \\
&-15519504x^4 + 11085360x^3 - 5116320x^2 + 1385670x - 167960)\big\}
\end{aligned}$$

Figure 2 shows the equilibrium that arises under a modest value for the Joy of Winning parameter, $c = \$1.00$.

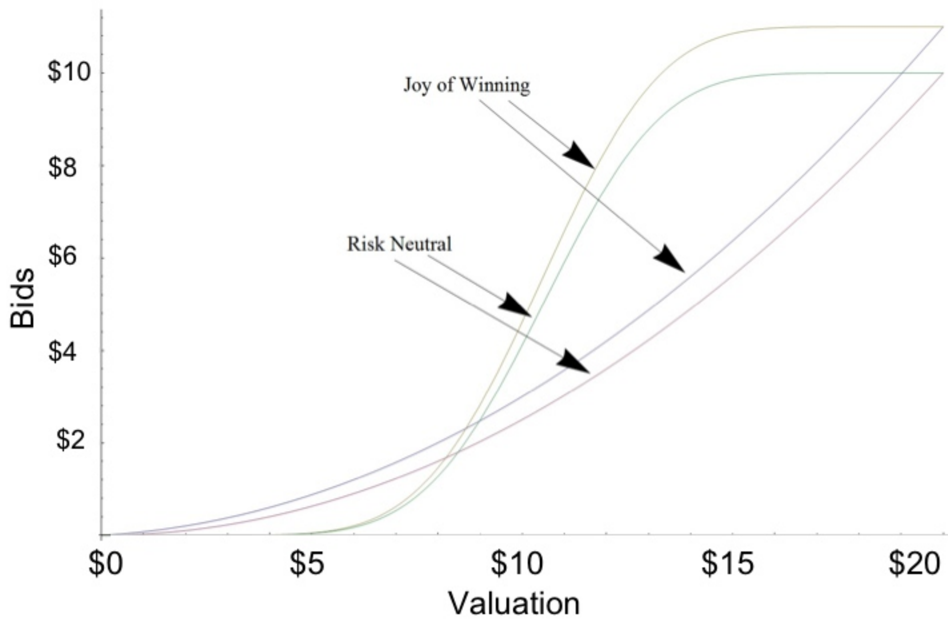

Figure 2: Joy of Winning Equilibrium

3 Subjects' Instructions

Before the experiment, subjects were given two sheets of paper to facilitate understanding of the experiment. The first page defined a list of terms that we would refer to throughout the experiment. The second page outlined logistics of the experiment with respect to how payments were determined. The two pages are found in the two following subsections.

3.1 Definitions

These terms were used throughout the experiment. We gave all subjects a hard copy reference sheet that they could use for clarification of the instructions.

> *Prize Value:* In each auction you will be assigned a prize value this is the amount of money that you will win if you receive a prize in that auction. So, if your prize value is $6.76 then you will receive $6.76 if you win that auction. If your prize value is $19.95 then you will receive $19.95 if you win that auction, and so on.

Bid: In each auction you must bid in order to win a prize. Prizes will be awarded to the participants with the highest bids. If there is one prize in the auction then the prize will go to the highest bidder. If there are ten prizes in the auction then they will go to the ten highest bidders, and so on.

> *NOTE:* You must pay your bid regardless of the outcome of the auction. Whether you win or you lose you will be charged your bid. If you bid \$5.34 in an auction and you lose then you will be charged \$5.34. Likewise, if you bid \$5.34 in an auction and you win then you will be charged \$5.34.

Payout: The amount of money you gain from an auction will depend on your bid and your prize value. If you win then you will receive your prize value and you will be charged your bid. If you lose then you will receive nothing and you will still be charged your bid.

> *EXAMPLE:* If you have a prize value of \$15.00 and you bid \$10.00 and you *win* the auction then you will receive your prize value, \$15.00, and you will be charged your bid, \$10.00, meaning that your payout will be (positive) \$5.00.
>
> *EXAMPLE:* If you have a prize value of \$15.00 and you bid \$10.00 and you *lose* the auction then you will receive nothing and you will be charged your bid, \$10.00, meaning that your payout will be (negative) -\$10.00.

Auction That Counts: After you have completed the experiment, one round will be selected at random to be the Auction That Counts. The Auction That Counts will determine part of your Take-Home Pay and your Payout from the Auction That Counts could be positive or negative. We select one round at random as the Auction That Counts so that no participant accumulates too many losses and has a negative Take-Home Pay. Also, we select the Auction That Counts randomly so that you take every round seriously because it could be the Auction That Counts.

Take-Home Pay: You will receive a $20.00 show-up fee for participating today. In addition to this show-up fee, we will combine your Payout from the Money Auction to your $20 show-up fee to determine your Take-Home Pay.

3.2 Instructions:

The page shown below explained the means by which we would calculate payments for each subject. All subjects were given a hard copy of this sheet for their reference.

<u>Experimental Instructions</u>

Thank you for volunteering for our experiment! Today you will participate in several different auctions in order to win money. The auctions will not be standard auctions, however, so there are some things that you need to keep in mind:

- ***Participation Payment:*** The payment for participating today is $20.
- ***Auctions:*** Today's auctions will be different from normal auctions. Like all auctions, you will bid money in order to win a prize and the prizes will go to the highest bidders. However, the number of bidders and the number of prizes will change. The first type of auction will have 2 people bidding for 1 prize. In these auctions the pairs will be randomly reassigned each round. The second type of auction will have 20 people bidding for 10 prizes.
- ***Bid***: In each auction you will specify a bid in order to win a prize. <u>*You must pay your bid regardless of the outcome of the auction*</u> so choose carefully. Whether you win or you lose you <u>*will*</u> be charged your bid.
    - ***EXAMPLE***: If you bid $5.34 in an auction and you lose, you will be charged $5.34. Likewise, if you bid $5.34 in an auction and you win, you will be charged $5.34.
- ***Winning Bid***: If there are more bids below your bid than above it, your bid is declared a Winning Bid and you will receive a prize. That is, if your bid is greater than half of the bids in the auction then you have a Winning Bid.
    - ***EXAMPLE***: In a 2-Person auction your bid must be higher than the other bid to be a Winning Bid. In a 20-Person auction your bid must be among the 10 highest bids to be a Winning Bid.
    - ***Ties***: The computer will randomly break ties between equal bids.
- ***Prize Value***: The "prizes" in these auctions will be worth a certain amount of money, and this amount will be different for each bidder. Every round you will be assigned a new "Prize Value." This Prize Value is the amount of money that you will win <u>*if*</u> you have a Winning Bid in the auction.
    - ***Possible Values:*** Prize Values are between $0.01 and $20.00 and every amount of money between them is equally likely. On average, half of the Prize Values will be above $10.00 and half of the Prize Values will be below $10.00 but the actual numbers will vary.
    - ***Hidden Values:*** You will see your Prize Value but will not see anyone else's Prize Value. Likewise, no one will see your Prize Value but you. All prize values will be drawn in the same way as yours.

3.3 Pre-Experiment Quiz:

Pre-Experiment Quiz

1. Suppose I have a prize value of *\$6.25* and I bid *\$5.00*
   a. If I **win** my payout will be:__________
   b. If I **lose** my payout will be:__________
2. Suppose I have a prize value of *\$12.12* and I bid *\$15.00*
   a. If I **win** my payout will be:__________
   b. If I **lose** my payout will be:__________
3. Suppose I am in an auction with 2 players and 1 prize. If I have a prize value of *\$10.50* and I bid *\$9.00* and my opponent bids *\$10.00.*
   a. My payout will be:___________
4. Suppose I am in an auction with 2 players and 1 prize. If I have a prize value of *\$7.25* and I bid *\$6.00* and my opponent bids *\$5.00*.
   a. My payout will be:___________
5. Suppose I am in an auction with 20 players and 10 prizes. If I have a prize value of *\$14.75* and I bid *\$8.00* and my opponents' bids are (in increasing order): *\$0 \$1.53 \$2.01 \$2.75 \$3.00 \$4.40 \$4.83 \$5.51 \$7.08 \$8.08 \$10.00 \$10.05 \$12.24 \$13.76 \$15.90 \$16.85 \$17.00 \$18.84 \$18.99*.
   a. My payout will be:___________
6. Suppose I am in an auction with 20 players and 10 prizes. If I have a prize value of *\$14.75* and I bid *\$8.00* and my opponents' bids are (in increasing order): *\$0.54 \$1.03 \$2.91 \$3.75 \$3.80 \$4.40 \$4.83 \$5.51 \$6.08 \$7.10 \$7.75 \$10.35 \$13.34 \$13.89 \$14.90 \$17.85 \$17.90 \$18.84 \$19.99*.
   a. My payout will be:___________
7. Suppose I have a prize value of \$3.00 and I bid \$6.00.
   a. The highest my payout can be is:_______________

8. Finally, consider a more complicated situation: Here I have bid in 2 different auctions:
    a. The first auction has 20 players and 10 prizes. If I have a prize value of *$14.75* and I bid *$8.00* and my opponents' bids are (in increasing order): *$0.54 $1.03 $2.91 $3.75 $3.80 $4.40 $4.83 $5.51 $6.08 $7.10 $7.75 $10.35 $13.34 $13.89 $14.90 $17.85 $17.90 $18.84 $19.99*.
    b. The second auction has 2 players and 1 prize. If I have a prize value of *$14.75* and I bid *$10.00* and my opponent bids *$5.00*.
    c. If you are offered the payout from the first auction plus *$1.50* your final payout will be: _________
    d. If you are offered the payout from the second auction your final payout will be: ________
    e. Lastly, if I offer you the payout from the outcome of the first auction plus *$1.50* or the payout from the second auction you are deciding between what two numbers? ________ & ________.

3.4 Auction Round Introduction

Prior to the auction round, we posted an introduction with instructions:

> Welcome to the auction round! In this round you will participate in 2 different auctions. The size of your prize has been randomly determined. We call it your "Prize Value" and you can see it below. This is the amount of money that you will receive if you win one of the prizes.
>
> In each auction, you will compete with other participants for prizes by bidding a certain amount of money. You will be required to pay this bid regardless of the outcome of the auction. Even if you fail to win one of the prizes, you must pay your bid.
>
> If there are more bids below your bid than above it, then you have a Winning Bid and will receive one of the prizes. That is, you will receive your Prize Value given below.
>
> If your bid is not a Winning Bid, then you will simply lose the amount that you bid in the auction.
>
> On the next 2 pages you will enter bids for both auctions. After all of the participants submit bids for both of the auctions, you will be told the outcomes of the auctions. You will not know the outcomes of the auctions until you have finished both auctions.
>
> **Your Prize Value $XX.XX**

3.5 Auction Round

Each screen of the auction round was identical, reminding the subject of instructions and important valuations:

> **This is the N-Person Auction**
>
> Remember:
>
> - You must pay your bid regardless of the outcome.
> - You will win your Prize Value if your bid is higher than $\frac{N}{2}$ opponents' bids.

- One auction will be selected randomly to be the Auction That Counts, so treat each auction as if it is the Auction That Counts.

On this page you have been randomly assigned to an auction with $N-1$ other participants and yourself. You are competing to win one of $\frac{N}{2}$ prizes.

- Number of participants: **N**
- Number of prizes: $\frac{N}{2}$

Your Prize Value $XX.XX
Please enter your bid here:

3.6 Auction Results

After the subjects all bid in both auctions, we posted all results on one page:

- Your Prize Value was: $XX.XX
- Your bid in the 2-Person Auction was: $XX.XX
- In the 2-Person Auction you were in group G and you (Won/Lost)
- Your Payout from the 2-Person Auction: $XX.XX

[List all 20 bids in all 10 auctions with winning bids paired with losing bids. The subject's bid is bold.]

- Your bid in the 20-Person Auction was: $XX.XX
- In the 20-Person Auction you (Won/Lost)
- Your Payout from the 20-Person Auction: $XX.XX

[List all 20 bids in the auction with winning bids in the top row and losing bids in the bottom row. The subject's bid is bold.]

3.7 Risk Aversion Elicitation

There are three different risk preference elicitation tasks in the style of Andreoni and Harbaugh (2010).

Remember:

- The computer will choose 1 of the 3 pages of lotteries to determine your payment.
- You will be paid according to the outcome of the lottery you selected.

Please decide which option you prefer the most. Indicate your preference by filling in the one button next to your most preferred option:

- Win $\$X_1$ with chance C_1 in 100

  ...

- Win $\$X_{10}$ with chance C_{10} in 100