

Reconciling Models of Diffusion and Innovation:
A Theory of the Productivity Distribution and Technology Frontier
Online Technical Appendix

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A Exogenous Innovation

This section derives the equilibrium when innovation is an exogenous process.

A.1 Normalization

Define the normalized distribution of productivity, as the distribution of productivity relative to the endogenous adoption threshold $M(t)$:

$$\Phi_i(t, Z) \equiv F_i(t, \log(Z/M(t))) \tag{A.1}$$

Differentiate to obtain the PDF yields

$$\partial_Z \Phi_i(t, Z) = \frac{1}{Z} \frac{\partial F_i(t, \log(Z/M(t)))}{\partial z} = \frac{1}{Z} \partial_z F_i(t, z) \tag{A.2}$$

¹References to equations, etc. in the main paper are prefixed by *Main*.

Differentiate (A.1) with respect to t and use the chain rule to obtain the transformation of the time derivative

$$\partial_t \Phi_i(t, Z) = \frac{\partial F_i(t, \log(Z/M(t)))}{\partial t} - \frac{M'(t)}{M(t)} \frac{\partial_i F(t, \log(Z/M(t)))}{\partial z} \quad (\text{A.3})$$

Use the definition $g(t) \equiv M'(t)/M(t)$ and the definition of z ,

$$\partial_t \Phi_i(t, Z) = \partial_t F_i(t, z) - g(t) \partial_z F_i(t, z) \quad (\text{A.4})$$

Normalizing the Law of Motion This is derived with a more general adoption process, where $\hat{F}_i(t, z)$ is the mass of agents who draw a technology below z with the innovation state i . In our baseline case, $\hat{F}_\ell(t, z) = F(t, z)$ and $\hat{F}_h(t, z) = 0$.

Substitute (A.2) and (A.4) into Main (7) and (8),

$$\begin{aligned} \frac{\partial F_\ell(t, \log(Z/M(t)))}{\partial t} &= g(t) \frac{\partial F_\ell(t, \log(Z/M(t)))}{\partial z} - \lambda_\ell F_\ell(t, \log(Z/M(t))) + \lambda_h F_h(t, \log(Z/M(t))) \\ &\quad + (S_\ell(t) + S_h(t)) \hat{F}_\ell(t, \log(Z/M(t))) - S_\ell(t) \\ &\quad - \eta F_\ell(t, \log(Z/M(t))) + \eta \mathbb{H} \left(\log(Z/M(t)) - \log(\bar{Z}(t)/M(t)) \right) \end{aligned} \quad (\text{A.5})$$

$$\begin{aligned} \frac{\partial F_h(t, \log(Z/M(t)))}{\partial t} &= g(t) \frac{\partial F_h(t, \log(Z/M(t)))}{\partial z} - \lambda_h F_h(t, \log(Z/M(t))) + \lambda_\ell F_\ell(t, \log(Z/M(t))) \\ &\quad - \gamma \frac{Z}{\bar{Z}} \frac{\partial F_h(t, \log(Z/M(t)))}{\partial z} + (S_\ell(t) + S_h(t)) \hat{F}_h(t, \log(Z/M(t))) - S_h(t) \\ &\quad - \eta F_h(t, \log(Z/M(t))) \end{aligned} \quad (\text{A.6})$$

Use the definition of z and reorganize to find the normalized KFEs,

$$\begin{aligned} \partial_t F_\ell(t, z) &= -\lambda_\ell F_\ell(t, z) + \lambda_h F_h(t, z) + g(t) \partial_z F_\ell(t, z) + S(t) \hat{F}_\ell(t, z) - S_\ell(t) \\ &\quad - \eta F_\ell(t, z) + \eta \mathbb{H}(z - \bar{z}(t)) \end{aligned} \quad (\text{A.7})$$

$$\partial_t F_h(t, z) = \lambda_\ell F_\ell(t, z) - \lambda_h F_h(t, z) + (g(t) - \gamma) \partial_z F_h(t, z) + S(t) \hat{F}_h(t, z) - S_h(t) - \eta F_h(t, z) \quad (\text{A.8})$$

where the domain of the normalized KFE at time t is $[0, \bar{z}(t)]$. Recall that the unnormalized flux (assuming $M'(t) \geq \gamma M(t)$) is

$$S_\ell(t) \equiv M'(t) \partial_Z \Phi_\ell(t, M(t)) \quad (\text{A.9})$$

$$S_h(t) \equiv \underbrace{(M'(t) - \gamma M(t))}_{\text{Relative Speed of Boundary}} \underbrace{\partial_Z \Phi_h(t, M(t))}_{\text{PDF at boundary}}. \quad (\text{A.10})$$

This is consistent with the solution to the ODEs in equations Main (7) and (8) at $Z = M(t)$. Normalizing and substituting from (A.2),

$$S_\ell(t) = g(t)\partial_z F_\ell(t, 0) \quad (\text{A.11})$$

$$S_h(t) = (g(t) - \gamma)\partial_z F_h(t, 0) \quad (\text{A.12})$$

Normalizing the Value Function Define the normalized value of the firm as,

$$v_i(t, \log(Z/M(t))) \equiv \frac{V_i(t, Z)}{M(t)} \quad (\text{A.13})$$

Rearrange and differentiate (A.13) with respect to t

$$\begin{aligned} \partial_t V_i(t, Z) &= M'(t)v_i(t, \log(Z/M(t))) - M'(t)\frac{\partial v_i(t, \log(Z/M(t)))}{\partial z} \\ &\quad + M(t)\frac{\partial v_i(t, \log(Z/M(t)))}{\partial t} \end{aligned} \quad (\text{A.14})$$

Divide by $M(t)$ and use the definition of $g(t)$

$$\frac{1}{M(t)}\partial_t V_i(t, Z) = g(t)v_i(t, z) - g(t)\partial_z v_i(t, z) + \partial_t v_i(t, z) \quad (\text{A.15})$$

Differentiate (A.13) with respect to Z and rearrange

$$\frac{1}{M(t)}\partial_Z V_i(t, Z) = \frac{1}{Z}\partial_z v_i(t, z) \quad (\text{A.16})$$

Divide Main (2) by $M(t)$ and then substitute from (A.15) and (A.16),

$$\begin{aligned} r\frac{1}{M(t)}V_h(t, Z) &= \frac{Z}{M(t)} + \gamma\frac{M(t)}{M(t)}\frac{Z}{Z}\partial_z v_h(t, z) + g(t)v_h(t, z) - g(t)\partial_z v_h(t, z) \\ &\quad + \lambda_h(v_\ell(t, z) - v_h(t, z)) + \frac{\eta}{M(t)}(V_\ell(t, \bar{Z}(t)) - V_h(t, Z)) + \partial_t v_h(t, z) \end{aligned} \quad (\text{A.17})$$

Use (A.13) and the definition of z and rearrange,

$$\begin{aligned} (r - g(t))v_h(t, z) &= e^z + (\gamma - g(t))\partial_z v_h(t, z) + \lambda_h(v_\ell(t, z) - v_h(t, z)) \\ &\quad + \eta(v_\ell(t, \bar{z}(t)) - v_h(t, z)) + \partial_t v_h(t, z) \end{aligned} \quad (\text{A.18})$$

Similarly, for Main (1)

$$(r - g(t))v_\ell(t, z) = e^z - g(t)\partial_z v_\ell(t, z) + \lambda_\ell(v_h(t, z) - v_\ell(t, z)) \\ + \eta(v_\ell(t, \bar{z}(t)) - v_\ell(t, z)) + \partial_t v_\ell(t, z) \quad (\text{A.19})$$

Optimal Stopping Conditions Recall that we are solving the general problem with draws to $\hat{\Phi}_i(t, Z)$ which nests our benchmark model. Divide the value-matching condition in Main (4) by $M(t)$,

$$\frac{V_i(t, M(t))}{M(t)} = \int_{M(t)}^{\bar{Z}(t)} \frac{V_\ell(t, Z)}{M(t)} \partial_Z \hat{\Phi}_\ell(t, Z) dZ + \int_{M(t)}^{\bar{Z}(t)} \frac{V_h(t, Z)}{M(t)} \partial_Z \hat{\Phi}_h(t, Z) dZ - \frac{M(t)}{M(t)} \zeta \quad (\text{A.20})$$

Use the substitutions in (A.2) and (A.13), and a change of variable $z = \log(Z/M(t))$ in the integral, which implies that $dz = \frac{1}{Z} dZ$. Note that the bounds of integration change to $[\log(M(t)/M(t)), \log(\bar{Z}(t)/M(t))] = [0, \bar{z}(t)]$

$$v_i(t, 0) = \int_0^{\bar{z}(t)} v_\ell(t, z) d\hat{F}_\ell(t, z) + \int_0^{\bar{z}(t)} v_h(t, z) d\hat{F}_h(t, z) - \zeta \quad (\text{A.21})$$

Evaluate (A.16) at $Z = M(t)$, and substitute this into Main (6) to give the smooth-pasting condition

$$\partial_z v_i(t, 0) = 0 \quad (\text{A.22})$$

A.2 Exogenous Markov Innovation

Rather than the log utility in the main paper, here we use a general firm discount rate r .

Proof of Main Proposition 1. Define the following to simplify notation,

$$\alpha \equiv (1 + \hat{\lambda}) \frac{S - \eta}{g} \quad (\text{A.23})$$

$$\hat{\lambda} \equiv \frac{\lambda_\ell}{\eta + \lambda_h} \quad (\text{A.24})$$

$$\bar{\lambda} \equiv \frac{r - g + \lambda_\ell + \lambda_h}{r - g + \lambda_h} \quad (\text{A.25})$$

$$\nu = \frac{r - g + \eta \bar{\lambda}}{g} \quad (\text{A.26})$$

Solve for $F_h(z)$ in Main (13),

$$F_h(z) = \hat{\lambda} F_\ell(z) \quad (\text{A.27})$$

Substitute this back into Main (12) to get an ODE in F_ℓ

$$0 = gF'_\ell(z) + (S - \eta)(1 + \hat{\lambda})F_\ell(z) + \eta\mathbb{H}(z - \bar{z}) - S \quad (\text{A.28})$$

Solve this ODE using $F_\ell(0) = 0$

$$F_\ell(z) = \begin{cases} \frac{S}{(S-\eta)(1+\hat{\lambda})}(1 - e^{-\alpha z}) & 0 \leq z < \bar{z} \\ \frac{S}{(S-\eta)(1+\hat{\lambda})}(1 - e^{-\alpha \bar{z}}) & z = \bar{z} \end{cases} \quad (\text{A.29})$$

This function is continuous at $z = \bar{z}$, and therefore so is $F_h(z)$. The unconditional distribution is,

$$F(z) = (1 + \hat{\lambda})F_\ell(\bar{z}) \quad (\text{A.30})$$

$$= \frac{S}{S - \eta}(1 - e^{-\alpha \bar{z}}) \quad (\text{A.31})$$

Use the boundary condition that $F(\bar{z}) = 1$, and solve for \bar{z} with the assumption that $S > \eta$,

$$\bar{z} = \frac{\log(S/\eta)}{\alpha} \quad (\text{A.32})$$

The PDF of the unconditional distribution is,

$$F'(z) = \frac{\alpha S}{S - \eta} e^{-\alpha z} \quad (\text{A.33})$$

To solve for the value, solve Main (19) for $v_h(z)$,

$$v_h(z) = \frac{e^z + (\lambda_h - \eta)v_\ell(z) + \eta v_\ell(\bar{z})}{r - g + \lambda_h} \quad (\text{A.34})$$

Substitute into Main (18) and simplify

$$(r - g + \eta)v_\ell(z) = e^z + \eta v_\ell(\bar{z}) - \frac{g}{\lambda} v'_\ell(z) \quad (\text{A.35})$$

Solve Main (21) with (A.35) and simplify,

$$v_\ell(z) = \frac{\bar{\lambda}}{g + (r + \eta - g)\bar{\lambda}} e^z + \frac{\eta}{r - g + \eta} v_\ell(\bar{z}) + \frac{1}{(r + \eta - g)(\nu + 1)} e^{-z\nu} \quad (\text{A.36})$$

Evaluate (A.36) at \bar{z} and solve for $v_\ell(\bar{z})$,

$$v_\ell(\bar{z}) = \left(-\frac{\eta}{g-r} + 1 \right) \left(\frac{e^{\bar{z}\bar{\lambda}}}{g + (\eta + r - g)\bar{\lambda}} + \frac{e^{-\nu\bar{z}}}{(\eta + r - g)(\nu + 1)} \right) \quad (\text{A.37})$$

Substitute (A.37) into (A.36) to find an expression for $v_\ell(z)$

$$v_\ell(z) = \frac{\bar{\lambda}}{g(1+\nu)} \left(e^z + \frac{1}{\nu} e^{-\nu z} + \frac{\eta}{r-g} \left(e^{\bar{z}} + \frac{1}{\nu} e^{-\nu\bar{z}} \right) \right) \quad (\text{A.38})$$

Substitute (A.33) and (A.38) into the value-matching condition in Main (20) and evaluate the integral,

$$\zeta + \frac{1}{r-g} = \frac{S\alpha\bar{\lambda} \left(-\frac{e^{-\nu\bar{z}}(-1+e^{-\alpha\bar{z}})\eta}{(-g+r)\alpha\nu} + \frac{e^{\bar{z}}\eta(e^{-\alpha\bar{z}}-1)}{\alpha(g-r)} + \frac{-e^{-(\alpha+\nu)\bar{z}+1}}{\nu(\alpha+\nu)} + \frac{-e^{\bar{z}-\alpha\bar{z}+1}}{\alpha-1} \right)}{g(S-\eta)(\nu+1)} \quad (\text{A.39})$$

To find an implicit equation for the equilibrium S , take (A.39) and substitute for α and \bar{z} from (A.23) and (A.32)

□

A.3 No Equilibrium Exist with $g < \gamma$ and $\eta > 0$

With the equilibrium in Main Proposition 1, while the $g \leq \gamma$ comes directly from assuming an initial condition with finite support, this section demonstrates that the $g < \gamma$ equilibrium do not exist for any $\eta > 0$.

Proof of Uniqueness of Main Proposition 1. In order to demonstrate that there are no stationary equilibrium with $g < \gamma$ and $\eta > 0$, we will do a proof by contradiction: (1) assume there is some constant $g < \gamma$ which is the consumer's optimal choice (i.e. balancing adoption costs and benefits); (2) use the constant g to calculate the expectation of the stationary productivity distribution; (3) show that the mean cannot be stationary relative to the adoption costs, and hence the $g(t)$ cannot have been an optimal choice.

Prior to finding the evolution of the moments, we need to be careful of exactly where agents are removed from the distribution, so replace the S in the CDF with the heaviside function removing at the threshold $\log(M(t)/M(t)) = 0$, i.e. $S\mathbb{H}(z)$. Take the normalized version in (A.7) and (A.8) and instead work in PDFs in order to later take a Laplace transformation. We maintain the assumption of a constant g and S with draws from the unconditional $F(t, \cdot)$. Use that the derivative of the Heaviside is the dirac-delta, $\delta(\cdot)$, to

find,

$$\partial_t f_\ell(t, z) = g\partial_z f_\ell(t, z) + (S - \lambda_\ell - \eta)f_\ell(t, z) + (S + \lambda_h)f_h(t, z) - S\delta(z) + \eta\delta(z - \bar{z}) \quad (\text{A.40})$$

$$\partial_t f_h(t, z) = (g - \gamma)\partial_z f_h(t, z) + \lambda_\ell f_\ell(t, z) - (\lambda_h + \eta)f_h(t, z) \quad (\text{A.41})$$

Here, after differentiating, we have the dirac-delta, $\delta(z)$, in (A.40) to remove those adopting agents at the threshold (which was normalized to $z = 0$), and the insertion of η arrival rate of agents at \bar{z} .

Taking inspiration from Gabaix et al. (2016), use the bilateral Laplace transform on the z variable to the new ξ space, such that $\mathcal{F}_i(t, \xi) \equiv \int_{-\infty}^{\infty} e^{-\xi z} f_i(t, z) dz$. Applying this transform to the ODEs in (A.40) and (A.41) gives,²

$$\partial_t \mathcal{F}_\ell(t, \xi) = g\xi \mathcal{F}_\ell(t, \xi) + (S - \lambda_\ell - \eta)\mathcal{F}_\ell(t, \xi) + (S + \lambda_h)\mathcal{F}_h(t, \xi) - S + \eta e^{-\bar{z}\xi} \quad (\text{A.42})$$

$$\partial_t \mathcal{F}_h(t, \xi) = (g - \gamma)\xi \mathcal{F}_h(t, \xi) + \lambda_\ell \mathcal{F}_\ell(t, \xi) - (\eta + \lambda_h)\mathcal{F}_h(t, \xi) \quad (\text{A.43})$$

From Gabaix et al. (2016) equation (16), evaluating at $\xi = -1$ are the moments of the $Z/M(t)$ distribution. Hence, to be a stationary first moment (for a given \bar{z}), substitute into a time-invariant (A.42) and (A.43) to find,

$$0 = -g\mathcal{F}_\ell(t, -1) + (S - \lambda_\ell - \eta)\mathcal{F}_\ell(t, -1) + (S + \lambda_h)\mathcal{F}_h(t, -1) - S + \eta e^{\bar{z}} \quad (\text{A.44})$$

$$0 = -(g - \gamma)\mathcal{F}_h(t, -1) + \lambda_\ell \mathcal{F}_\ell(t, -1) - (\eta + \lambda_h)\mathcal{F}_h(t, -1) \quad (\text{A.45})$$

Solve this algebraic system of equations for $\mathcal{F}_\ell(t, -1)$ and $\mathcal{F}_h(t, -1)$ and then use the linearity of the Laplace transform to find $\mathcal{F}(t, -1) = \mathcal{F}_\ell(t, -1) + \mathcal{F}_h(t, -1)$,

$$\mathbb{E}_t [Z/M(t)] = \mathcal{F}(t, -1) = \frac{g - \gamma + \eta + \lambda_h + \lambda_\ell}{(g - S + \eta)(g - \gamma + \eta + \lambda_h) - \lambda_\ell(S - g + \gamma - \eta)} (\eta e^{\bar{z}} - S) \quad (\text{A.46})$$

Since $g < \gamma$ by assumption, from (D.40), $\lim_{t \rightarrow \infty} \bar{z}(t) = \infty$. Therefore, (A.46) diverges for any $\eta > 0$, proving that the mean of the distribution cannot be stationary if $\bar{z} \rightarrow \infty$.

To finish the proof by contradiction, recall that the change of variables to $z \equiv \log(Z/M(t))$ was already normalized relative to $M(t)$, and hence is normalized relative to the adoption cost, $\zeta M(t)$. Furthermore, since $v_\ell(z) > z$, Main (20) cannot hold with equality at all

²Most of the transformation comes through using the linearity of the operator, and the general formula that the bilateral Laplace transform of a derivative. That is, using a simple notation: $\mathcal{L}\{f'(z)\} = \xi \mathcal{F}(\xi)$. The other important formula is that $\mathcal{L}\{\delta(z - c)\} = e^{-c\xi}$, and $\mathcal{L}\{\delta(z)\} = 1$

points. Therefore, we have the contradiction that $M(t)$ leading to the g cannot have been optimal. \square

A.4 Common Adoption Threshold for All Idiosyncratic States

Proof. This section derives sufficient conditions for heterogeneous firms to choose the same adoption threshold. It is kept as general as possible to nest both the exogenous and endogenous versions of the model.

Allow for some discrete type i , and augment the state of the firm with an additional state x (which could be a vector or a scalar). Assume that there is some control u that controls the infinitesimal generator \mathbb{Q}_u of the Markov process on type i and, potentially, x .³ Also assume that the agent can control the growth rate $\hat{\gamma}$ at some cost. The feasibility set of the controls is $(u, \hat{\gamma}) \in U(t, i, z, x)$. The cost of the controls for adoption and innovation have several requirements for this general property to hold: (a) the net value of searching, $v_s(t)$, is identical for all types i , productivities z , and additional state x , (b) the minimum of the cost function is 0 and in the interior of the feasibility set: $\min_{(\hat{\gamma}, u) \in U(t, i, z, x)} c(t, z, \hat{\gamma}, i, x, u) = 0$, for all t, x, i ; and (c) the value of a jump to the frontier, $\bar{v}(t)$, is identical for all agent states (e.g. $\bar{v}(t) = v_\ell(t, \bar{z}(t)) = v_h(t, \bar{z}(t))$).⁴

Let flow profits be potentially type-dependent, $\pi_i(t, z, x)$, but require that $\pi(t, 0) \equiv \pi_i(t, 0, x)$ is identical for all i and x . Then, the normalization of the firm's problem gives the following set of necessary conditions:

$$(r - g(t))v_i(t, z, x) = \max_{(\hat{\gamma}, u) \in U(\cdot)} \left\{ \pi_i(t, z, x) - c(t, z, \hat{\gamma}, i, x, u) + (\hat{\gamma} - g) \frac{\partial v_i(t, z, x)}{\partial z} + \frac{\partial v_i(t, z, x)}{\partial t} + \mathbf{e}_i \cdot \mathbb{Q}_u \cdot v(t, z, x) + \eta(\bar{v}(t) - v_i(t, z, x)) \right\} \quad (\text{A.47})$$

$$v_i(t, \underline{z}(t, i, x), x) = v_s(t) \quad (\text{A.48})$$

$$\frac{\partial v_i(t, \underline{z}(t, i, x), x)}{\partial z} = 0, \quad (\text{A.49})$$

where $\underline{z}(t, i, x)$ is the normalized search threshold for type i and additional state x . To prove that these must be identical, we will assume that $\underline{z}(t, i, x) = 0$ for all types and

³Ordering the states as $\{l, h\}$, the infinitesimal generator for this continuous-time Markov chain is $\mathbb{Q} = \begin{bmatrix} -\lambda_\ell & \lambda_\ell \\ \lambda_h & -\lambda_h \end{bmatrix}$, with adjoint operator \mathbb{Q}^* . The KFE and Bellman equations can be formally derived using these operators and the drift process.

⁴Without this requirement, firms may have differing incentives to “wait around” for arrival rates of jumps at the adoption threshold. A slightly weaker requirement is if the arrival rates and value are identical only at the threshold: $\eta(t, 0, \cdot)$ and $\bar{v}(t, 0, \cdot)$ are idiosyncratic states.

additional states, and show that this leads to identical necessary optimal stopping conditions. Evaluating at $z = 0$,

$$v_i(t, 0, x) = v_s(t) \tag{A.50}$$

$$\frac{\partial v_i(t, 0, x)}{\partial z} = 0. \tag{A.51}$$

Note that equations (A.50) and (A.51) are identical for any i and x . Substitute equations (A.50) and (A.51) into equation (A.47) to obtain

$$(r - g(t))v_s(t) = \max_{(\hat{\gamma}, u) \in U(\cdot)} \{ \pi(t, 0) - c(t, z, \hat{\gamma}, i, x, u) + \mathbf{e}_i \cdot \mathbb{Q}_u \cdot v_s(t) + \eta(\bar{v}(t) - v_s(t)) + v'_s(t) \}. \tag{A.52}$$

Since in order to be a valid intensity matrix, all rows in \mathbb{Q}_u add to 0 for any u , the last term is 0 for any i or control u ,

$$(r - g(t))v_s(t) = \max_{(\hat{\gamma}, u) \in U(\cdot)} \{ \pi(t, 0) - c(t, z, \hat{\gamma}, i, x, u) + v'_s(t) + \eta(\bar{v}(t) - v_s(t)) \}. \tag{A.53}$$

The optimal choice for any i or x is to minimize the costs of the $\hat{\gamma}$ and u choices. Given that $\hat{\gamma}$ only shows up as a cost, and our assumption that the cost at the minimum is 0 and is interior,

$$(r - g(t))v_s(t) = \pi(t, 0) + v'_s(t) + \eta(\bar{v}(t) - v_s(t)) \quad \text{for all } i. \tag{A.54}$$

Therefore, the necessary conditions for optimal stopping are identical for all i, x, z , confirming our guess. Furthermore, equation (A.54) provides an ODE for $v_s(t)$ based on aggregate $g(t)$ and $\bar{v}(t)$ changes. Solving this in a stationary environment gives an expression for v_s in terms of equilibrium g, \bar{v} and the common $\pi(0)$,

$$v(0) \equiv v_s = \frac{\pi(0) + \eta\bar{v}}{r - g + \eta} \tag{A.55}$$

Furthermore, note from equation Main (30) that $\pi(0) = 1$ for all variations of the model in the body of the paper.

□

A.5 Limit of Draw Arrival Process

The following is a heuristic derivation of the law of motion and cost function for searching which yields a conditional draw above the firm's search threshold.

Instead of instantaneous draws, assume a firm choosing to adopt has an arrival rate of

$\bar{\lambda} > 0$ of opportunities. While attempting to adopt, they pay a normalized flow cost of $\zeta\bar{\lambda}$. Note that we are scaling the flow cost by the arrival rate in order to take the limit and have a finite expected value of search costs.

Furthermore, assume that the firm draws *unconditionally* from the z distribution in the economy (rather than simply those above their current threshold), and starts with a ℓ type. In the stationary equilibrium, as all agents start low, searching firms accept a draw if they get above the normalized cutoff of 0. The proof will construct a limit where agents get a successful draw above 0 in any infinitesimal time period, and hence draw from the conditional distribution of $z \geq 0$.

Define $F_{\ell\lambda}(z)$ and $F_{h\lambda}(z)$ to be the CDFs of agents in the $z < 0$ region. As firms in the region $F_{\ell\lambda}(z)$ are otherwise identical, define the mass of searching agents as $F_{\ell\lambda}(0)$. Assume that agents have an unconditional draw of all z within the economy, then conditional on a draw, the probability of escaping the $F_{\ell\lambda}(0)$ mass is $(1 - F_{\ell\lambda}(0) - F_{h\lambda}(0))$. It is easily shown that the arrival rate of *successful* draws is then $\bar{\lambda}(1 - F_{\ell\lambda}(0) - F_{h\lambda}(0))$. The distribution of waiting times until the first success is an exponential distribution with this parameter. The survivor function is therefore: $e^{-\bar{\lambda}(1 - F_{\ell\lambda}(0) - F_{h\lambda}(0))t}$. Due to the total mass of one, $F_{\ell\lambda}(0) + F_{h\lambda}(0) \in (0, 1)$, so the survivor function is decreasing in t . $F_{\ell\lambda}(0)$ is independent of the $\bar{\lambda}$ arrival rate when taking limits as no agents enter this region from successful searches. Taking the limit for any t , $\lim_{\bar{\lambda} \rightarrow \infty} e^{-\bar{\lambda}(1 - F_{\ell\lambda}(0) - F_{h\lambda}(0))t} = 0$. Therefore, in the limit in equilibrium, $F_{\ell\lambda}(0) = 0$ as measure 1 agents get a successful draw in any strictly positive interval. The same arguments can be used to explain why $F_{h\lambda}(0) = 0$.

To ensure that the expected search costs in this limit are finite, calculate the present discounted value of flow payments until the first success. This is the exponential distribution with parameter $\bar{\lambda}(1 - F_{\ell\lambda}(0) - F_{h\lambda}(0))$ and flow cost $\zeta\bar{\lambda}$

$$\mathbb{E}[\text{search costs}] = \int_0^\infty \left(\int_0^t \zeta\bar{\lambda}e^{-rs} ds \right) \bar{\lambda}(1 - F_{\ell\lambda}(0) - F_{h\lambda}(0))e^{-\bar{\lambda}(1 - F_{\ell\lambda}(0) - F_{h\lambda}(0))t} dt \quad (\text{A.56})$$

$$= \frac{\bar{\lambda}\zeta}{r + \bar{\lambda}(1 - F_{\ell\lambda}(0) - F_{h\lambda}(0))} \quad (\text{A.57})$$

Take the limit and use the result that $F_{\ell\lambda}(0)$ and $F_{h\lambda}(0)$ converge to show

$$\lim_{\bar{\lambda} \rightarrow \infty} \mathbb{E}[\text{search costs}] = \zeta \quad (\text{A.58})$$

Therefore, in the limit the model can have draws directly from above the current threshold, with measure 0 remaining behind, and a cost for an instantaneous adoption of ζ . Since the flow of adopters is of measure 0, whether the draw is from the conditional or unconditional distribution is irrelevant.

B Endogenous Markov Innovation

This builds on the previous section to add additional derivations for endogeneity. For brevity, any equations that remain identical will be left out of the discussion.

Keep in mind that two things that change with endogeneity are the growth rate in the innovation state, i.e. $\gamma \rightarrow \gamma(\cdot)$, and the profits function due to licensing, i.e., $e^z \rightarrow \pi(z)$.

B.1 Stationary BGP with Endogeneity

Proof of Main Propositions 3 and 4 . To create a stationary solution for the value function define a change of variables,⁵

$$w_i(z) \equiv e^{-z}v'_i(z) \tag{B.1}$$

From (34) and (35),

$$w_\ell(0) = w_h(0) = 0 \tag{B.2}$$

Differentiate (B.1) and reorganize ,

$$e^{-z}v''_i(z) = w'_i(z) + w_i(z) \tag{B.3}$$

Take the first order necessary condition of the Hamilton-Jacobi-Bellman equation in Main (33), and reorganize

$$\gamma(z) = \frac{\chi}{2}e^{-z}v'_h(z) \tag{B.4}$$

Substitute this back into Main (33) to get a non-linear ODE,

$$(r - g)v_h(z) = \pi(z) - gv'_h(z) + \frac{\chi}{4}e^{-z}v'_h(z)^2 + \lambda_h(v_\ell(z) - v_h(z)) + \eta(v_\ell(\bar{z}) - v_h(z)) \tag{B.5}$$

Differentiate Main (32),

$$(r - g)v'_\ell(z) = \pi'(z) - gv''_\ell(z) + \lambda_\ell(v'_h(z) - v'_\ell(z)) - \eta v'_\ell(z) \tag{B.6}$$

⁵Our approach is to normalize and then substitute the FOC of the HJBE into the Bellman equation to form a nonlinear ODE, which we can solve numerically using collocation methods as discussed in the Computational Appendix. An alternative approach to solving the HJBE numerically might be to use upwind finite difference methods as in Achdou et al. (2017).

Multiply (B.6) by e^{-z} and use (B.1) and (B.3) and Main (42).

$$(r + \lambda_\ell + \eta - (1 - \psi)gF'(0))w_\ell(z) = 1 - gw'_\ell(z) + \lambda_\ell w_h(z) \quad (\text{B.7})$$

Note that using (B.3),

$$e^{-z}\partial_z \left(e^{-z}v'_h(z)^2 \right) = 2e^{-z}v''_h(z)e^{-z}v'_h(z) - \left(e^{-z}v'_h(z) \right)^2 \quad (\text{B.8})$$

$$= 2w_h(z)w'_h(z) + w_h(z)^2 \quad (\text{B.9})$$

Differentiate (B.5), multiply by e^{-z} , and use (B.1), (B.3) and (B.9) and Main (42)

$$(r + \lambda_h + \eta)w_h(z) = 1 - \left(g - \frac{\chi}{2}w_h(z) \right) w'_h(z) + (\lambda_h + (1 - \psi)gF'(0))w_\ell(z) + \frac{\chi}{4}w_h(z)^2 \quad (\text{B.10})$$

From (B.4),

$$\gamma(z) = \frac{\chi}{2}w_h(z) \quad (\text{B.11})$$

$$g \equiv \frac{\chi}{2}w_h(\bar{z}) \quad (\text{B.12})$$

□

Proof of Main Proposition 4. For the case where $\eta \rightarrow 0$, and $\bar{z} \rightarrow \infty$, we can check the asymptotic value comes from (B.1)

$$\lim_{z \rightarrow \infty} w_i(z) = c_i \quad (\text{B.13})$$

To find an upper bound on g , note that as $w_i(z)$ is increasing, the maximum growth rate is as $\bar{z} \rightarrow \infty$. In the limit, $\lim_{z \rightarrow \infty} w'_i(z) = 0$ as $w_i(z)$ have been constructed to be stationary. Furthermore, note that the maximum g from (B.12) is,

$$g = \lim_{\bar{z} \rightarrow \infty} \frac{\chi}{2}w_h(\bar{z}) = \frac{\chi}{2}c_h \quad (\text{B.14})$$

Therefore, looking at the asymptotic limit of (B.7) and (B.10),

$$(r + \lambda_\ell + \eta - (1 - \psi)\frac{\chi}{2}c_h F'(0))c_\ell = 1 + \lambda_\ell c_h \quad (\text{B.15})$$

$$(r + \lambda_h + \eta)c_h = 1 + \left(\lambda_h + (1 - \psi)\frac{\chi}{2}c_h F'(0) \right) c_\ell + \frac{\chi}{4}c_h^2 \quad (\text{B.16})$$

Given a $F'(0)$, (B.15) and (B.16) provide a quadratic system of equations c_ℓ and c_h —and ultimately g through (B.14). While analytically tractable given an $F'(0)$, this quadratic has

a complicated solution—except if $\psi = 0$. For that case, define

$$\bar{\lambda} \equiv \frac{r + \eta + \lambda_\ell + \lambda_h}{r + \eta + \lambda_\ell} \quad (\text{B.17})$$

Then, an upper bound on the growth rate with $\psi = 1$ and $\eta > 0$ is

$$g < \bar{\lambda}(r + \eta) \left[1 - \sqrt{1 - \frac{\chi}{\bar{\lambda}(r + \eta)^2}} \right] \quad (\text{B.18})$$

where if $\eta = 0$, the unique solution is,

$$g = \bar{\lambda}r \left[1 - \sqrt{1 - \frac{\chi}{\bar{\lambda}r^2}} \right] \quad (\text{B.19})$$

where a necessary condition for an interior equilibrium is

$$r > \sqrt{\frac{\chi}{\bar{\lambda}}} \quad (\text{B.20})$$

□

B.2 Bargaining Derivation

Proof of Main Proposition 2 . From standard Nash bargaining, with a total surplus value of $v_\ell(z)$, let \hat{v} be the proportion of the surplus obtained by the licensee and $v_\ell(z) - \hat{v}$ be the proportion obtained by the licensor. As is apparent in (B.22), if $\psi = 1$, then the technology is adopted for free and the licensee gains the entire value such that $\hat{v}(z) = v_\ell(z)$.

The timing is that the adopting firm first pays the adoption cost and then, upon the realization of the match, negotiations over licensing commence.

$$\arg \max_{\hat{v}} \left\{ (\hat{v} - v(0))^\psi (v_\ell(z) - \hat{v})^{1-\psi} \right\}. \quad (\text{B.21})$$

Solving for the surplus split, the value for a licensee that matches a firm with productivity z is,

$$\hat{v}(z) = (1 - \psi)v(0) + \psi v_\ell(z), \quad (\text{B.22})$$

while the licensor gains

$$v_\ell(z) - \hat{v}(z) = (1 - \psi)(v_\ell(z) - v(0)). \quad (\text{B.23})$$

There is an equal probability of adopting from any licensor and a unit measure of firms.

Thus, the flow of adopters engaging any licensing firm is just the flow of adopters $S = S_\ell + S_h$,

$$\pi(z) = e^z + \underbrace{gF'_\ell(0) + (g - \gamma(0))F'_h(0)}_{\text{Amount of Licenses}} \underbrace{(1 - \psi)(v_\ell(z) - v(0))}_{\text{Profits per Licensee}}. \quad (\text{B.24})$$

Since the surplus split does not introduce state dependence to the cost of adopting, the smooth-pasting condition is unchanged. However, since adopters may not gain the full surplus from the newly adopted technology due to licensing costs, the value-matching condition takes bargaining into account. Adapting Main (20), the value-matching condition is

$$v(0) = \frac{1}{\rho} = \int_0^\infty \underbrace{[\psi v_\ell(z) + (1 - \psi)v(0)]}_{\text{Surplus with licensing}} dF(z) - \zeta. \quad (\text{B.25})$$

Rearranging, the value-matching condition is identical to that previously derived in Main (20), but with a proportional increase in the adoption cost,

$$v(0) = \frac{1}{\rho} = \int_0^\infty v_\ell(z) dF(z) - \frac{\zeta}{\psi}. \quad (\text{B.26})$$

□

C Example Parameter Values

See Table 1 for a summary of parameters used in examples.

We set $\gamma = 0.02$ to target a 2 percent growth rate and $\rho = 0.01$ to generate a real interest rate of 3 percent.

Transition rates λ_h and λ_ℓ are chosen to roughly match firms' growth-rates, with firms growing faster than 5 percent annually labeled h types and all other firms labeled ℓ -types. While the transition rates are sensitive to the h -threshold growth rate, all resulting numerical analysis is unchanged by wide variation in this threshold, as all calibrated transition rates are high enough to suggest that there is little persistence in either state and that the process essentially acts like iid growth rates.

Where appropriate, we target an approximate tail index of $\alpha = 2.12$, which corresponds to a tail parameter of 1.06 in the size or profits distribution used in Luttmer (2007) with a typical markup. Since the model is not sensitive to the tails, this is not a sensitive parameter but provides some discipline when jointly choosing the adoption and innovation costs.

We use $\bar{z} = 1.61$ (i.e., the frontier to minimum productivity ratio is 5). This ratio is larger than the $\bar{z} = 0.651$ (1.92 ratio in levels) documented by Syverson (2011) between the top and bottom decile within narrowly defined industries, and is closer to the 5:1 ratio found

| Parameters | Value/Target | Calibration |
|-------------------------------|--------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $\{\lambda_\ell, \lambda_h\}$ | $\{0.533074, 1.12766\}$ | Matches estimation of 2 state Markov transition matrix for firm growth rates using Compustat with firms in SIC 2000-3999. Model is not sensitive to these parameters, as long as they are not too low |
| ψ | $[0.5, 1.0]$ with 0.95 | Matches median 5% royalties of large firms reported from RoyaltySource in Kemmerer and Lu (2012) |
| ρ | 0.01 | Target interest rate $r = .03$ when $g = 0.02$ |
| $\{\chi, \zeta\}$ | $g = 0.02$ and $\alpha = 2.12$ | Targets 2% growth rate, and an underlying tail parameter of the firm size distribution of 1.06 (which translates to $\alpha = 2.12$ using the rough adjustment implied by monopolistic competition). Note: The growth rates are a function of ψ and other parameters which are calibrated separately. |
| $\{\bar{z}, \eta\}$ | $\bar{z} \in [0.651, \infty]$ | If $\eta \approx 0$, then \bar{z} is set large enough for numerical stability to approximate ∞ (keeping in mind that $e^{\bar{z}}$ is the actual multiplier on productivity of the frontier, so $\bar{z} = 3.0$ translates to a ratio of productivity of the frontier to the threshold of 20.1.) |

Table 1: Summary of Parameter Values

in Hsieh and Klenow (2009) in India and China. The qualitative results are not strongly driven by this fact, as we look across various \bar{z} level.

D Equilibria without Leap-frogging (i.e. $\eta = 0$)

The assumption of a small probability of adopting the frontier technology, $\eta > 0$, or taking the limit $\eta \rightarrow 0$, removes many of the difficulties associated with hysteresis associated with this model. This self-contained section looks at the equilibria in the case of no leap-frogging (i.e. $\eta = 0$).

D.1 Equilibrium with $g > \gamma$

If $\Phi(0, Z)$ has infinite-support, the normalized $F(t, Z)$ will converge to a stationary distribution as $t \rightarrow \infty$. A continuum of stationary distributions exists; they are determined by initial conditions (i.e. hysteresis), and each is associated with its own aggregate growth rate. To characterize the continuum of stationary distributions, parameterize the set of solutions by a scalar α . By construction, α will be the tail index of the unconditional distribution $F(z)$.

$$\text{Let } \vec{F}(z) \equiv \begin{bmatrix} F_\ell(z) \\ F_h(z) \end{bmatrix}.$$

Define $\mathbf{0}, \mathbf{1}, \mathbf{I}$ as a vector of 0, 1, and the identity matrix and the following:

$$A \equiv \begin{bmatrix} \frac{1}{g} \\ \frac{1}{g-\gamma} \end{bmatrix} \quad B \equiv \begin{bmatrix} \frac{r+\lambda_\ell-g}{g} & -\frac{\lambda_\ell}{g} \\ -\frac{\lambda_h}{g-\gamma} & \frac{r+\lambda_h-g}{g-\gamma} \end{bmatrix} \quad (\text{D.1})$$

$$C \equiv \begin{bmatrix} \frac{gF'_\ell(0)+(g-\gamma)F'_h(0)-\lambda_\ell}{g} & \frac{\lambda_h}{g} \\ \frac{\lambda_\ell}{g-\gamma} & \frac{gF'_\ell(0)+(g-\gamma)F'_h(0)-\lambda_h}{g-\gamma} \end{bmatrix} \quad D \equiv \begin{bmatrix} F'_\ell(0) \\ F'_h(0) \end{bmatrix} \quad (\text{D.2})$$

$$\vec{F}(z) \equiv \begin{bmatrix} F_\ell(z) \\ F_h(z) \end{bmatrix} \quad v(z) \equiv \begin{bmatrix} v_\ell(z) \\ v_h(z) \end{bmatrix} \quad (\text{D.3})$$

Proposition 1 (Stationary Equilibrium with Infinite Support and $g > \gamma$). *There exists a continuum of equilibria parameterized by α , which, by construction, is the tail index of F . An equilibrium is determined by the $g(\alpha)$ that satisfies*

$$\frac{1}{\rho} + \zeta = \int_0^\infty \left[\left[(\mathbf{I} + B)^{-1} \left(e^{\mathbf{I}z} + e^{-Bz} B^{-1} \right) A \right]^T e^{-Cz} D \right] dz \quad (\text{D.4})$$

and the parameter restrictions given in (D.23) to (D.26). The stationary distributions and the value functions are given by:

$$\vec{F}(z) = (\mathbf{I} - e^{-Cz}) C^{-1} D \quad (\text{D.5})$$

$$\vec{F}'(z) = e^{-Cz} D \quad (\text{D.6})$$

$$\vec{v}(z) = (\mathbf{I} + B)^{-1} \left(e^{\mathbf{I}z} + e^{-Bz} B^{-1} \right) A. \quad (\text{D.7})$$

Proof. For ease of exposition, in this section, we model the adoption technology as firms copying both the type and productivity of their draw from the unconditional distribution.⁶ That is, the normalized adoption is no longer to become ℓ and draw from the unconditional distribution. With this, the value-matching and KFEs become

$$v(0) = \frac{1}{\rho} = \underbrace{\int_0^\infty v_\ell(z) dF_\ell(z) + \int_0^\infty v_h(z) dF_h(z)}_{\text{Adopt both } i \text{ and } Z \text{ of draw}} - \zeta \quad (\text{D.8})$$

$$0 = gF'_\ell(z) - \lambda_\ell F_\ell(z) + \lambda_h F_h(z) + (S_\ell + S_h)F_\ell(z) - S_\ell \quad (\text{D.9})$$

$$0 = (g - \gamma)F'_h(z) - \lambda_h F_h(z) + \lambda_\ell F_\ell(z) + (S_\ell + S_h)F_h(z) - S_h. \quad (\text{D.10})$$

⁶While an exactly correlated draw of the type and the productivity is not necessary here, you can proof that independent draws for adopters of Z and the innovation type i have infinite-support equilibria only with degenerate stationary distributions. The finite-support cases do not impose the same requirements.

Then the equilibrium conditions can be written as a linear set of ODEs:

$$v'(z) = Ae^z - Bv(z) \quad (\text{D.11})$$

$$v'(0) = \mathbf{0} \quad (\text{D.12})$$

$$\vec{F}'(z) = -C\vec{F}(z) + D \quad (\text{D.13})$$

$$\vec{F}(0) = \mathbf{0} \quad (\text{D.14})$$

$$\vec{F}(\infty) \cdot \mathbf{1} = 1 \quad (\text{D.15})$$

$$v_\ell(0) = v_h(0) = \int_0^\infty (v(z)^T \cdot \vec{F}'(z)) dz - \zeta \quad (\text{D.16})$$

Solve these as a set of matrix ODEs, where e^{Az} is a matrix exponential. Start with (D.11) and (D.12) to get,⁷

$$v(z) = (I + B)^{-1} (e^{Iz} + e^{-Bz} B^{-1}) A. \quad (\text{D.19})$$

Evaluate at $z = 0$,

$$v(0) = B^{-1} A = \left[\frac{1}{r-g} \quad \frac{1}{r-g} \right]. \quad (\text{D.20})$$

Then (D.13) and (D.14) gives

$$\vec{F}(z) = (\mathbf{I} - e^{-Cz}) C^{-1} D. \quad (\text{D.21})$$

Take the derivative,

$$\vec{F}'(z) = e^{-Cz} D. \quad (\text{D.22})$$

For (D.19) and (D.21) to be well defined as $z \rightarrow \infty$, we have to impose parameter restrictions that constrain the growth rate g so that the eigenvalues of B and C are positive or have

⁷The equation $\vec{F}'(z) = A\vec{F}(z) + b$ subject to $\vec{F}(0) = \mathbf{0}$ has the solution,

$$\vec{F}(z) = (e^{Az} - \mathbf{I}) A^{-1} b. \quad (\text{D.17})$$

The derivation of these results uses that $\int_0^T e^{tA} dt = A^{-1} (e^{TA} - \mathbf{I})$. With appropriate conditions on eigenvalues, this implies that $\int_0^\infty e^{tA} dt = -A^{-1}$.

Equations of the form, $v'(z) = Ae^z - B \cdot v(z)$ with the initial condition $v'(0) = \mathbf{0}$ have the solution,

$$v(z) = (I + B)^{-1} (e^{Iz} + e^{-Bz} B^{-1}) A. \quad (\text{D.18})$$

This derivation exploits commutativity, as both e^{Bz} and $(I + B)^{-1}$ can be expanded as power series of B .

positive real parts. S_l and S_h are defined in equations Main (16) and (17) in terms of $F'_l(0)$ and $F'_h(0)$, C and B will have roots with positive real parts iff their determinant and their trace are strictly positive. For C it is straightforward to compute that the conditions for a positive trace and determinant are

$$S_l + S_h > \frac{(g - \gamma) \lambda_l + g \lambda_h}{(g - \gamma) + g} \quad (\text{D.23})$$

$$S_h + S_l > \lambda_h + \lambda_l \quad (\text{D.24})$$

and for B the corresponding conditions are

$$r > g > \gamma \quad (\text{D.25})$$

$$r - g + \lambda_h + \lambda_l > 0. \quad (\text{D.26})$$

With these conditions imposed, we can proceed to characterize the solutions to the value functions and the stationary distribution.

Evaluate (D.22) at $z = 0$,

$$\vec{F}'(0) = D. \quad (\text{D.27})$$

Take the limit of (D.21)

$$\vec{F}(\infty) = C^{-1}D \quad (\text{D.28})$$

and (D.15) becomes

$$\mathbf{1} = C^{-1}D \cdot \mathbf{1}. \quad (\text{D.29})$$

We can check that, by construction, with the C and D defined by (D.2), (D.29) is fulfilled for any $F'_\ell(0)$, $F'_h(0)$, λ_ℓ , and λ_h .

For $\vec{F}'(z)$ to define a valid PDF it is necessary for $\vec{F}'(z) > 0$ for all z . It can be shown, that for $C > 0$, the only $D > 0$ fulfilling this requirement is one proportional to the eigenvector associated with the dominant eigenvalue of C .⁸ The unique constant of proportionality is

⁸Since $C > 0$ and irreducible (in this case off diagonals not zero), then by Perron-Frobenius it has a simple dominant real root α and an associated eigenvector $\nu > 0$. Hence, as $\vec{F}(0) = 0$, $F_\ell(\infty) + F_h(\infty) = 1$, and $\vec{F}'(z) > 0$, we have a valid PDF. This uniqueness of the ν solution only holds if the other eigenvector of C has a positive and negative coordinate, which always holds in our model.

determined by (D.29). The two eigenvectors of C fulfilling this proportionality are,

$$\nu_i \equiv \begin{bmatrix} \frac{F'_h(0) \left(\gamma(g-\gamma)F'_h(0) \pm \sqrt{2(g-\gamma)\lambda_l \left(\gamma \left(g \left(F'_h(0) + F'_l(0) \right) - \gamma F'_h(0) \right) + g\lambda_h \right) + \left(\gamma \left(g \left(F'_h(0) + F'_l(0) \right) - \gamma F'_h(0) \right) - g\lambda_h \right)^2 + (g-\gamma)^2 \lambda_l^2 + \gamma g F'_l(0) - g\lambda_h + g\lambda_l - \gamma\lambda_l} \right)}{2g\lambda_l} \\ F'_h(0) \end{bmatrix} \quad (\text{D.30})$$

Denote ν as the eigenvector with both positive elements—which is associated with the dominant eigenvalue—then as discussed above, $D \propto \nu$. Use (D.2) and (D.30), and note the eigenvector has already been normalized to match the second parameter.

$$D = \nu. \quad (\text{D.31})$$

The 2nd coordinate already holds with equality by construction, for the first coordinate equating (D.2) and (D.30). Equate the first parameter and choose the positive eigenvector,

$$F'_\ell(0) = \frac{F'_h(0) \left(\gamma(g-\gamma)F'_h(0) - \sqrt{2(g-\gamma)\lambda_l \left(\gamma \left(g \left(F'_h(0) + F'_l(0) \right) - \gamma F'_h(0) \right) + g\lambda_h \right) + \left(\gamma \left(g \left(F'_h(0) + F'_l(0) \right) - \gamma F'_h(0) \right) - g\lambda_h \right)^2 + (g-\gamma)^2 \lambda_l^2 + \gamma g F'_l(0) - g\lambda_h + g\lambda_l - \gamma\lambda_l} \right)}{2g\lambda_l} \quad (\text{D.32})$$

Solve this equation for $F'_\ell(0)$ and choose the positive root

$$F'_\ell(0) = \frac{F'_h(0)\lambda_h}{\gamma F'_h(0) + \lambda_\ell}. \quad (\text{D.33})$$

We can check that with the C and D defined by (D.2), (D.29) is fulfilled by construction. The value-matching in (D.16) becomes,

$$\frac{1}{r-g} + \zeta = \int_0^\infty \left[\left[(I+B)^{-1} \left(e^{Iz} + e^{-Bz} B^{-1} \right) A \right]^T e^{-Cz} D \right] dz. \quad (\text{D.34})$$

Note that if B has positive eigenvalues, then $\lim_{z \rightarrow \infty} v(z) = (1+B)^{-1} (e^z) A$. Therefore, as long as C has a minimal eigenvalue (defined here as α), strictly greater than one, the integral is defined.

The tail index of the unconditional distribution, $F(z) \equiv F_\ell(z) + F_h(z)$ can be calculated from the C matrix in (D.22). As sums of power-law variables inherit the smallest tail index, the endogenous power-law tail is minimum eigenvalue of C . After the substitution for $F'_\ell(0)$ from above, the smallest eigenvalue of C is

$$\alpha \equiv \frac{((g-\gamma)F'_h(0) - \lambda_l) (\gamma(g-\gamma)F'_h(0) + g(\lambda_h + \lambda_l) - \gamma\lambda_l)}{g(g-\gamma) (\gamma F'_h(0) + \lambda_l)}. \quad (\text{D.35})$$

Solve (D.35) for $F'_h(0)$ as a function of α ,

$$F'_h(0) = \frac{g \left(\alpha\gamma - \lambda_h + \sqrt{(\lambda_h - \alpha\gamma)^2 + 2\lambda_l(\alpha\gamma + \lambda_h) + \lambda_l^2 - \lambda_l} \right) + 2\gamma\lambda_l}{2\gamma(g - \gamma)}. \quad (\text{D.36})$$

Substitute for $F'_i(0)$ into C and D to get a function in terms of g and α ,

$$C = \begin{bmatrix} \frac{-\alpha\gamma + 2\alpha g + \lambda_h + \sqrt{(\lambda_h - \alpha\gamma)^2 + 2\lambda_l(\alpha\gamma + \lambda_h) + \lambda_l^2 - \lambda_l}}{2g} & \frac{\lambda_h}{g} \\ \frac{\lambda_l}{g - \gamma} & \frac{-\alpha\gamma + 2\alpha g - \lambda_h + \sqrt{(\lambda_h - \alpha\gamma)^2 + 2\lambda_l(\alpha\gamma + \lambda_h) + \lambda_l^2 + \lambda_l}}{2(g - \gamma)} \end{bmatrix} \quad (\text{D.37})$$

$$D = \begin{bmatrix} \frac{\lambda_h \left(g \left(\alpha\gamma - \lambda_h + \sqrt{(\lambda_h - \alpha\gamma)^2 + 2\lambda_l(\alpha\gamma + \lambda_h) + \lambda_l^2 - \lambda_l} \right) + 2\gamma\lambda_l \right)}{\gamma g \left(\alpha\gamma - \lambda_h + \sqrt{(\lambda_h - \alpha\gamma)^2 + 2\lambda_l(\alpha\gamma + \lambda_h) + \lambda_l^2 + \lambda_l} \right)} \\ \frac{g \left(\alpha\gamma - \lambda_h + \sqrt{(\lambda_h - \alpha\gamma)^2 + 2\lambda_l(\alpha\gamma + \lambda_h) + \lambda_l^2 - \lambda_l} \right) + 2\gamma\lambda_l}{2\gamma(g - \gamma)} \end{bmatrix} \quad (\text{D.38})$$

While both $F'_i(0)$ could conceivably parameterize a set of solutions for each g , they are constrained by the eigenvector proportionality condition, which ensures that the manifold of solutions is 1 dimensional. \square

To interpret the expressions: (D.5) shows that the positivity of the tail index α is now equivalent to C having positive eigenvalues. For the decomposition of the option value, (D.7) shows that positive eigenvalues of B ensure the option values in the vector $v(z)$ converges to 0 as z increases.

In this section, the stationary distribution is a vector $\vec{F}(z)$ solving (D.9) and (D.10), a system of linear ODEs. If we define the unconditional distribution $F(z) \equiv F_\ell(z) + F_h(z)$, and if both $F_\ell(z)$ and $F_h(z)$ are power-laws, any mixture of these distributions inherits the smallest (i.e. thickest) tail parameter (as discussed in Gabaix (2009)). Since there are now two dimensions of heterogeneity, the tail index, α , is defined as that of the unconditional distribution, $F(z)$. The ODE solution for the vector $\vec{F}(z)$ given in Proposition 1 by (D.5) will depend on the roots of C (both positive). The smallest root of C , representing the slower rate of decay for both elements of $F(z)$, is the tail index α by the construction.

In Proposition 1 the initial condition $F'(0)$ is a vector, so in principle this raises the possibility that the continuum of stationary equilibria could be two dimensional, parametrized by $F'_\ell(0) > 0$ and by $F'_h(0) > 0$. However as shown this is not possible since the only initial condition that ensures that $F_\ell(z)$ and $F_h(z)$ remain positive and satisfy (D.9) and (D.10) is exactly the eigenvector of C corresponding to its dominant (Frobenius) eigenvalue. Since the eigenvector is determined only up to a multiplicative constant, the continuum of stationary distributions is therefore one dimensional. We use the smallest eigenvalue of C , defined as the tail index α , to solve for $F'_h(0)$, which then determines $F'_\ell(0)$ from the eigenvector re-

striction. This then allows us to obtain the expressions for C and D in terms of parameters, α and g . Then value-matching, (D.16) and (D.34), gives us expression (D.4) to define g in terms α , so we end up with a continuum of stationary equilibria parametrized by α .

D.2 Equilibrium with $g \leq \gamma$

With the infinite support initial condition, $g > \gamma$ was a possibility. Here we show that either infinite or finite support initial conditions can also lead to $g \leq \gamma$ (including the $\eta \rightarrow 0$ limit we take which leads to the $g = \gamma$ case).

Crucially here, the $g < \gamma$ equilibrium exist with both the infinite and finite initial conditions, which demonstrates that while an assumption of infinite support can lead to “unreasonable” growth rates above the maximum firm innovation rate, it does not need to. For the $g < \gamma$ equilibria, starting from an infinite or finite support initial condition is irrelevant.

The weak bound on $g \leq \gamma$ is easy to see when starting from a finite initial support. Assume the support is finite, assume by contradiction there exists an optimally chosen $g > \gamma$ and $\bar{Z} < \infty$. Note that $M(t) = M(0)e^{gt}$ and $\bar{Z}(t) = \bar{Z}(0)e^{\gamma t}$, so for any initial conditions $M(T) > \bar{Z}(T)$ for all t greater than some T . That is, the distribution compresses until it is a point at which point the minimum and maximum of support switch.

Proposition 2 (Equilibrium with $g < \gamma$). *For both a infinite- and finite-support initial conditions, there exists a continuum of equilibrium parameterized by $\alpha > 1$, which, by construction, is the tail index of the productivity distribution. An equilibrium is determined by the aggregate growth rate $g(\alpha) < \gamma$ that satisfies*

$$\frac{1}{\rho} + \zeta = a_2 b_1 \left(\frac{a_3}{\alpha + \beta_2} - \frac{a_3 b_2}{\bar{\alpha} + \beta_2} - \frac{1}{\alpha + \beta_1} + \frac{b_2}{\bar{\alpha} + \beta_1} \right) + a_1 b_1 \left(\frac{1}{1 - \alpha} + \frac{b_2}{\bar{\alpha} - 1} \right) \quad (\text{D.39})$$

given $a_1, a_2, a_3, b_1, b_2, \beta_1, \beta_2$, and $\bar{\alpha}$ defined in (D.44), (D.49) to (D.51) and (D.53) to (D.55).

Parameter restrictions are given in (D.57), (D.61) and (D.62).

Proof. Unlike the case with a infinite-support and $g > \gamma$ in Proposition 1, we can use the simple case of draws are from the unconditional distribution and all agents ended up ℓ (as in our baseline $\eta > 0$ model).

Given our $g < \gamma$ assumption, $S_h = 0$ as no agents in the high state will cross the barrier. For the support of the distribution, the minimum of support has been normalized to $\log(M(t)/M(t)) = 0$. The maximum of support starts at $\log(\bar{Z}(0)/M(0)) \equiv \bar{z}(0)$ and grows as $\bar{z}(t) = \log\left(\bar{z}(0)\frac{e^{\gamma t}}{e^{gt}}\right)$ or

$$\bar{z}(t) = \bar{z}(0) + (\gamma - g)t \quad (\text{D.40})$$

Hence, asymptotically, $\lim_{t \rightarrow \infty} \bar{z}(t) = \infty$ given our $\gamma > g$ assumption. Alternatively, if $\bar{Z}(0) = \infty$, then this solution fully nests the $g < \gamma$ case as well. Following the notation of Proposition 1 as much as possible, define

$$A \equiv \begin{bmatrix} \frac{1}{g} \\ \frac{1}{\gamma-g} \end{bmatrix} \quad B \equiv \begin{bmatrix} \frac{r+\lambda_\ell-g}{g} & -\frac{\lambda_\ell}{g} \\ -\frac{\lambda_h}{\gamma-g} & \frac{r+\lambda_h-g}{\gamma-g} \end{bmatrix} \quad (\text{D.41})$$

$$C \equiv \begin{bmatrix} f_\ell(0) - \frac{\lambda_\ell}{g} & f_\ell(0) + \frac{\lambda_h}{g} \\ -\frac{\lambda_\ell}{\gamma-g} & \frac{\lambda_h}{\gamma-g} \end{bmatrix} \quad D \equiv \begin{bmatrix} F'_\ell(0) \\ 0 \end{bmatrix} \quad (\text{D.42})$$

Otherwise, the $\vec{F}(z)$, and $v(z)$ are identical to those in (D.1) and (D.2), and the ODEs for the KFE and the value function fulfill the same (D.11) to (D.15).⁹ The only change in A and B from the previous definition was to swap the order to $\gamma - g$. Under these definitions, the matrix A and the eigenvalues of B are positive.

As we have unconditional draws ending up with the ℓ type, the value-matching condition is replaced with the standard one, as in Main (20)

$$v_\ell(0) = v_h(0) = \int_0^\infty v_\ell(z) F'(z) dz - \zeta \quad (\text{D.43})$$

The other matrices and generic solutions to the ODEs for the $v(z)$ and $F(z)$ are identical to those in (D.19), (D.21) and (D.22).

In order to better parameterize the solutions, reorganize the algebra to be in terms of the asymptotic tail parameter of the distribution, α , instead of $F'_\ell(0)$. Using this, there may be a continuum of $\{g, \alpha\}$ which generate stationary distributions (and have an accompanying $F'_\ell(0)$). Define the following,

$$\bar{\alpha} \equiv \frac{\lambda_h}{\gamma-g} - \frac{\lambda_\ell}{g} - \frac{\alpha\gamma\lambda_\ell}{g(\lambda_\ell + \lambda_h - \alpha(\gamma-g))} \quad (\text{D.44})$$

$$F'_\ell(0) \equiv \alpha \left(1 - \frac{\gamma\lambda_\ell}{g(-(\gamma-g)\alpha + \lambda_h + \lambda_\ell)} \right) \quad (\text{D.45})$$

$$= \alpha + \bar{\alpha} - \frac{\lambda_h}{\gamma-g} + \frac{\lambda_\ell}{g} \quad (\text{D.46})$$

Here, $\{\alpha, \bar{\alpha}\}$ are the eigenvalues of the matrix C , ordered so that $\bar{\alpha} > \alpha$

⁹Unlike the infinite horizon case with $g > \gamma$, we no longer require an argument based on Perron-Frobenius in the proof. The reason is that $F'_h(0) = 0$ trivially, so the manifold of the solution in $\{g, F'_\ell(0)\}$ is already of the correct dimension.

Take the solution (D.22) and expand using the constant definitions above,

$$\begin{aligned} \vec{F}'(z) &= \frac{\alpha \left(\lambda_h - \left(\frac{\gamma}{g} - 1 \right) \lambda_\ell - (\gamma - g) \alpha \right)}{(\gamma - g) (\bar{\alpha} - \alpha)} \frac{1}{\lambda_h + \lambda_\ell - \alpha(\gamma - g)} \\ &\times \begin{bmatrix} (\lambda_h - \alpha(\gamma - g)) e^{-\alpha z} - (\lambda_h - \bar{\alpha}(\gamma - g)) e^{-\bar{\alpha} z} \\ \lambda_\ell (e^{-\alpha z} - e^{-\bar{\alpha} z}) \end{bmatrix} \end{aligned} \quad (\text{D.47})$$

The solution for the unconditional PDF, $F'(z) \equiv F'_\ell(z) + F'_h(z)$, is,

$$F'(z) = b_1 \left(e^{-\alpha z} - b_2 e^{-\bar{\alpha} z} \right) \quad (\text{D.48})$$

Where,

$$b_1 \equiv \frac{\alpha \left(-\alpha(\gamma - g) + \left(1 - \frac{\gamma}{g} \right) \lambda_l + \lambda_h \right)}{(\bar{\alpha} - \alpha)(\gamma - g)} \quad (\text{D.49})$$

$$b_2 \equiv \frac{\bar{\alpha}(g - \gamma) + \lambda_h + \lambda_l}{\alpha(g - \gamma) + \lambda_h + \lambda_l} \quad (\text{D.50})$$

To solve the value function in (D.21), first define the value β_1 and β_2 as,

$$\beta_{1/2} \equiv -\frac{-g(\gamma - \lambda_h + \lambda_l) \pm \sqrt{(g \lambda_h + (\gamma - g) \lambda_l + \gamma(r - g))^2 + 4g(g - \gamma)(g - r)(g - \lambda_h - \lambda_l - r) + \gamma(\lambda_l + r)}}{2g(g - \gamma)} \quad (\text{D.51})$$

Using the eigenvectors and eigenvalues, the matrix exponential in (D.19) can be written through a standard eigendecomposition with the eigenvalues β_1, β_2 and associated eigenvectors. Use this technique and rearrange (D.19) to find,

$$v_\ell(z) = a_1 e^z + a_2 \left(e^{-\beta_1 z} - a_3 e^{-\beta_2 z} \right) \quad (\text{D.52})$$

where,

$$a_1 \equiv \frac{\gamma - 2g + \lambda_h + \lambda_l + r}{-2g(\lambda_l + r) + r(\gamma + \lambda_h + r) + \lambda_l(\gamma + r)} \quad (\text{D.53})$$

$$a_2 \equiv \frac{(-\beta_2 \gamma + (\beta_2 - 1)g + r) \left((g - \gamma) \lambda_h \lambda_l + g(-\gamma + 2g - \lambda_h - r)(g(\beta_1 - 1) + \lambda_h - \gamma\beta_1 + r) \right)}{(g - \gamma)(r - g) \lambda_h (\beta_1 - \beta_2) \left(-2g(\lambda_l + r) + r(\gamma + \lambda_h + r) + \lambda_l(\gamma + r) \right)} \quad (\text{D.54})$$

$$a_3 \equiv \frac{(g(\beta_1 - 1) - \gamma\beta_1 + r) \left((g - \gamma) \lambda_h \lambda_l + g(-\gamma + 2g - \lambda_h - r)(-\beta_2 \gamma + (\beta_2 - 1)g + \lambda_h + r) \right)}{(-\beta_2 \gamma + (\beta_2 - 1)g + r) \left((g - \gamma) \lambda_h \lambda_l + g(-\gamma + 2g - \lambda_h - r)(g(\beta_1 - 1) + \lambda_h - \gamma\beta_1 + r) \right)} \quad (\text{D.55})$$

To finalize the solution, for any given α within the set of admissible parameters, substitute

(D.48) and (D.52) into (D.43) and solve as an implicit function of g . Substitute and simplify to find,

$$\frac{1}{r-g} = a_2 b_1 \left(\frac{a_3}{\alpha + \beta_2} - \frac{a_3 b_2}{\bar{\alpha} + \beta_2} - \frac{1}{\alpha + \beta_1} + \frac{b_2}{\bar{\alpha} + \beta_1} \right) + a_1 b_1 \left(\frac{1}{1-\alpha} + \frac{b_2}{\bar{\alpha}-1} \right) - \zeta \quad (\text{D.56})$$

For a given α (or, equivalently, a g), this implicit equation provides a solution for the corresponding g given all of the g and α dependent constants in (D.44), (D.49) to (D.51) and (D.53) to (D.55).

Parameter Restrictions A set of parameter restrictions are required to ensure that $F'(z) > 0$ and that the eigenvalues of both B and C are strictly positive to ensure a non-explosive root.

To ensure that the PDF is positive at every point in (D.48), note that since $\bar{\alpha} > \alpha$, the term $0 < \frac{\lambda_\ell + \lambda_h - (\gamma - g)\bar{\alpha}}{\lambda_\ell + \lambda_h - (\gamma - g)\alpha} < 1$ is positive as long as $\lambda_\ell + \lambda_h - (\gamma - g)\bar{\alpha} > 0$. Similarly, for $F'_\ell(z) > 0$ in (D.47), a requirement is that $0 < \frac{\lambda_h - (\gamma - g)\bar{\alpha}}{\lambda_h - (\gamma - g)\alpha} < 1$ and $\lambda_h - (\gamma - g)\bar{\alpha} > 0$, which reduce to

$$g > \gamma - \frac{\lambda_\ell + \lambda_h}{\alpha} \quad (\text{D.57})$$

Note that the thicker the tail (i.e. smaller α), the greater the range of possible g to match the α .

For (D.21) to be well defined as $z \rightarrow \infty$, we have to impose parameter restrictions that constrain the growth rate g so that the eigenvalues of C are positive or have positive real parts. Note that,

$$\det \{C\} = \frac{\lambda_h + \lambda_\ell}{\gamma - g} f_\ell(0) > 0 \quad (\text{D.58})$$

And,

$$\text{Trace} \{C\} = f_\ell(0) + \frac{\lambda_h}{\gamma - g} - \frac{\lambda_\ell}{g} \quad (\text{D.59})$$

Hence, since the determinant is positive, a necessary condition for C to have two positive eigenvalues is for

$$f_\ell(0) > \frac{\lambda_\ell}{g} - \frac{\lambda_h}{\gamma - g} \quad (\text{D.60})$$

From (D.22) we see that $\vec{F}'(0) = D$ and $f_h(0) = 0$. If C does have two positive eigenvalues

lues, then from (D.21), $\vec{F}(\infty) = \frac{1}{\lambda_\ell + \lambda_h} [\lambda_h \quad \lambda_\ell]$, which fulfills $F(\infty) = 1$.

With the new constants, the condition in (D.60) to ensure positive eigenvalues becomes,

$$\alpha < \sqrt{\frac{(\lambda_h + \lambda_l)(g(\lambda_h + \lambda_l) - \gamma\lambda_l)}{g(g - \gamma)^2}} \quad (\text{D.61})$$

Furthermore, as α has been constructed to be the smallest eigenvalue, and hence is the tail parameter, a necessary condition to have a finite mean is that $\alpha > 1$. From this, the g in equilibrium must fulfill,

$$2g(\lambda_h + \lambda_l) > \sqrt{4g^2(g - \gamma)^2 + \gamma^2\lambda_l^2} + \gamma\lambda_l \quad (\text{D.62})$$

To ensure that the eigenvalues of B are positive, and hence $v_i(z)$ is well-defined, note that the determinant of B is always positive, and the trace of B requires $g < \frac{\gamma(r + \lambda_\ell)}{\gamma - \lambda_h + \lambda_\ell}$. However, in practice this upper-bound is greater than γ .

To summarize: the key conditions parameter restrictions are, (D.57), (D.61) and (D.62) □

As an example of this equilibrium, Figure 1 plots the equilibrium growth rate as a function of α . For smaller α values, the growth rate approaches $\gamma = 0.02$

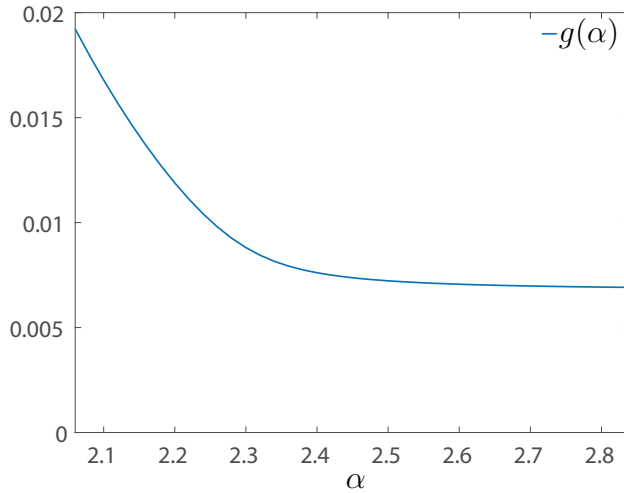


Figure 1: Exogenous $g < \gamma$ examples, for various $\alpha > 1$

In addition to the $g < \gamma$ equilibrium, consider the $g = \gamma$ which cannot be nested since the ODE becomes a DAE when $g = \gamma$. This is also the unique, limiting solution as $\eta \rightarrow 0$.

Proposition 3 (Stationary Equilibrium with $g = \gamma$ and $\eta = 0$). *There exists a unique maximum growth equilibrium with $g = \gamma$ and $\bar{z} \rightarrow \infty$. The unique maximum g stationary*

distribution is,

$$F_\ell(z) = \frac{1}{1+\hat{\lambda}} \left(1 - e^{-\alpha z}\right) \quad (\text{D.63})$$

$$F_h(z) = \hat{\lambda} F_\ell(z), \quad (\text{D.64})$$

where α is the tail index of the power-law distribution:

$$\alpha \equiv (1 + \hat{\lambda}) F'_\ell(0), \quad (\text{D.65})$$

and $F'_\ell(0)$ is determined explicitly by model parameters:

$$F'_\ell(0) = \frac{\lambda_h \left(\zeta(\rho+\gamma)((\rho+\gamma)+\lambda_h+\lambda_\ell) - \sqrt{\zeta\left(\left(4\gamma+(\rho+\gamma)^2\zeta\right)(-\gamma+(\rho+\gamma)+\lambda_h)^2 + 2(-2\gamma+(\gamma-(\rho+\gamma))(\rho+\gamma)\zeta)(\gamma-(\rho+\gamma)-\lambda_h)\lambda_\ell + (\gamma-(\rho+\gamma))^2\zeta\lambda_\ell^2\right) + \zeta\gamma^2 2 + \zeta(-\gamma)(3(\rho+\gamma)+2\lambda_h+\lambda_\ell)}{\gamma^2\zeta(\lambda_h+\lambda_\ell)(\gamma-(\rho+\gamma)-\lambda_h)} \quad (\text{D.66})$$

The firm value functions are,

$$v_\ell(z) = \frac{\bar{\lambda}}{\gamma + \rho\bar{\lambda}} e^z + \frac{1}{\rho(\nu + 1)} e^{-\nu z} \quad (\text{D.67})$$

$$v_h(z) = \frac{e^z + \lambda_h v_\ell(z)}{\rho + \lambda_h}, \quad (\text{D.68})$$

where ν is defined as

$$\nu \equiv \frac{\rho\bar{\lambda}}{\gamma} > 0. \quad (\text{D.69})$$

Proof. For the $g < \gamma$, see the nested solution in Proposition 2. Define the following to simplify notation,

$$\alpha \equiv (1 + \hat{\lambda}) \frac{S}{g} \quad (\text{D.70})$$

$$\hat{\lambda} \equiv \frac{\lambda_\ell}{\lambda_h} \quad (\text{D.71})$$

$$\bar{\lambda} \equiv \frac{\lambda_\ell}{r - g + \lambda_h} + 1 \quad (\text{D.72})$$

$$\nu = \frac{(r - g)\bar{\lambda}}{g} \quad (\text{D.73})$$

Take Main (13) at $\eta = 0$ and solve for $F_h(z)$

$$F_h(z) = \hat{\lambda} F_\ell(z) \quad (\text{D.74})$$

Substitute into Main (12) at $\eta = 0$

$$S = gF'_\ell(z) + (\hat{\lambda} + 1)SF_\ell(z) \quad (\text{D.75})$$

Solve this as an ODE in $F_\ell(z)$, subject to the $F_\ell(0) = 0$ boundary condition

$$F_\ell(z) = \frac{1}{1+\bar{\lambda}} (1 - e^{-\alpha z}) \quad (\text{D.76})$$

We can check that if $\alpha > 0$ the right boundary conditions hold

$$\lim_{z \rightarrow \infty} (F_\ell(z) + F_h(z)) = 1 \quad (\text{D.77})$$

Differentiate (D.76),

$$F'_\ell(z) = \frac{\alpha}{1+\bar{\lambda}} e^{-\alpha z} \quad (\text{D.78})$$

With (D.74), the PDF for the unconditional distribution, $F(z)$,

$$F'(z) = \alpha e^{-\alpha z} \quad (\text{D.79})$$

Solve Main (19) for $v_h(z)$ with $\eta = 0$

$$v_h(z) = \frac{e^z + \lambda_h v_\ell(z)}{r - g + \lambda_h} \quad (\text{D.80})$$

Substitute into Main (12) to find an ODE in $v_\ell(z)$

$$(r - g)v_\ell(z) = e^z + \lambda_h \hat{\lambda} \left(-v_\ell(z) + \frac{e^z + \lambda_h v_\ell(z)}{r - g + \lambda_h} \right) - gv'_\ell(z) \quad (\text{D.81})$$

Use the constant definitions and simplify

$$(r - g)v_\ell(z) = e^z - \frac{gv'_\ell(z)}{\bar{\lambda}} \quad (\text{D.82})$$

Solve this ODE subject to the smooth-pasting condition in Main (22) and simplify,

$$v_\ell(z) = \frac{\bar{\lambda}}{g + (r - g)\bar{\lambda}} e^z + \frac{1}{(r - g)(\nu + 1)} e^{-z\nu} \quad (\text{D.83})$$

Use the definitions of the constants and (D.83)

$$v_\ell(0) = \frac{1}{r - g} \quad (\text{D.84})$$

Substitute (D.79), (D.83) and (D.84) into the value-matching condition in Main (20) and simplify

$$\frac{1}{r-g} = \int_0^\infty \left[\frac{e^{z(\bar{\lambda}-\alpha-\frac{r\bar{\lambda}}{g})}\alpha g}{(g-r)(-r\bar{\lambda}+g(\bar{\lambda}-1))} + \frac{e^{z-z\alpha}\alpha\bar{\lambda}}{g+r\bar{\lambda}-g\bar{\lambda}} \right] dz - \zeta \quad (\text{D.85})$$

Evaluate the integral,

$$\zeta = \frac{\alpha(-r\bar{\lambda}+g(\bar{\lambda}-\alpha+1))}{(g-r)(r\bar{\lambda}+g(\alpha-\bar{\lambda}))(\alpha-1)} - \frac{1}{(r-g)(\nu+1)} - \frac{\bar{\lambda}}{g+r\bar{\lambda}-g\bar{\lambda}} \quad (\text{D.86})$$

Substitute for α gives an implicit equation in S

$$0 = \zeta + \frac{g\left(\frac{1}{r-g} + \frac{\bar{\lambda}}{S-g+S\bar{\lambda}} - \frac{\bar{\lambda}}{S-g\bar{\lambda}+r\bar{\lambda}+S\bar{\lambda}}\right)}{-r\bar{\lambda}+g(\bar{\lambda}-1)} + \frac{1}{(r-g)(\nu+1)} \quad (\text{D.87})$$

As $g = \gamma$ in equilibrium, only S is unknown. This equation is a quadratic in S , and can be written analytically in terms of model parameters as,

$$S = \frac{\lambda_h \left(\zeta r(r+\lambda_h+\lambda_\ell) - \sqrt{\zeta((4g+r^2\zeta)(-g+r+\lambda_h)^2 + 2(-2g+(g-r)r\zeta)(g-r-\lambda_h)\lambda_\ell + (g-r)^2\zeta\lambda_\ell^2) + \zeta g^2 2 + \zeta(-g)(3r+2\lambda_h+\lambda_\ell)} \right)}{2\zeta(\lambda_h+\lambda_\ell)(g-r-\lambda_h)} \quad (\text{D.88})$$

From this S , α can be calculated through (D.70), and the the rest of the equilibrium follows. \square

A final proof is to show that there do not exist any equilibria with a finite support where the \bar{z} does not diverge. Here we generalize the draw distribution to be $F(z)^\kappa$ for $\kappa > 0$ (where the $\kappa = 1$ is our baseline) in order to demonstrate that this is not a knife-edge result due to direct, unconditional draws from the distribution.

Proposition 4 (No Equilibria with Finite \bar{z} when $\eta = 0$). *Generalize the model to support draws from a distorted unconditional distribution, i.e. $F(z)^\kappa$ for $\kappa > 0$. For any κ , if $\eta = 0$ there do not exist any equilibria where the steady-state $\bar{z} < \infty$.*

Proof. First, note that if $g < \gamma$, then \bar{z} trivially diverges. Consider the $g = \gamma$ case and define

the following to simplify notation,

$$\alpha \equiv (1 + \hat{\lambda}) \frac{S}{g} \quad (\text{D.89})$$

$$\hat{\lambda} \equiv \frac{\lambda_\ell}{\lambda_h} \quad (\text{D.90})$$

$$\bar{\lambda} \equiv \frac{\lambda_\ell}{r - g + \lambda_h} + 1 \quad (\text{D.91})$$

$$\nu = \frac{(r - g)\bar{\lambda}}{g} \quad (\text{D.92})$$

Reorganizing the KFEs,

$$F_h(z) = \hat{\lambda} F_\ell(z) \quad (\text{D.93})$$

Substitute into Main Paper (12) with $\eta = 0$ to form a first-order non-linear ODE in $F_\ell(z)$

$$0 = S(1 + \hat{\lambda})^\kappa F_\ell(z)^\kappa + g F'_\ell(z) - S \quad (\text{D.94})$$

Rearrange,

$$F'_\ell(z) = \frac{S}{g} - \frac{S}{g} (1 + \hat{\lambda})^\kappa F_\ell(z)^\kappa \quad (\text{D.95})$$

This non-linear ODE is separable,

$$dz = \frac{dF_\ell(z)}{\frac{S}{g} - \frac{S}{g} (1 + \hat{\lambda})^\kappa F_\ell(z)^\kappa} \quad (\text{D.96})$$

Integrate,

$$z + C_1 = \int_0^{F_\ell} \frac{1}{\frac{S}{g} - \frac{S}{g} (1 + \hat{\lambda})^\kappa q^\kappa} dq \quad (\text{D.97})$$

Define the following function of $q \in [0, 1]$, where ${}_1F_2(\cdot)$ is the Hypergeometric function.¹⁰

¹⁰One definition for the Hypergeometric function is derived by simplifying Euler's representation in its integral form, is,

$$\int_0^y \frac{1}{a + b\tilde{y}^\kappa} d\tilde{y} = \frac{y}{a} {}_1F_2(1, 1/\kappa, 1 + 1/\kappa; -\frac{b}{a}y^\kappa) \quad (\text{D.98})$$

The derivatives fulfill,

$$\frac{d}{dp} {}_1F_2(a, b, c; p) = \frac{ab}{c} {}_1F_2(a + 1, b + 1, c + 1; p) \quad (\text{D.99})$$

$$Q_\ell(q) = \frac{g}{S} q {}_1F_2\left(1, 1/\kappa, 1 + 1/\kappa; (1 + \hat{\lambda})^\kappa q^\kappa\right) \quad (\text{D.100})$$

Assume that $Q_\ell(q)$ has an inverse $Q_\ell^{-1}(\cdot)$. Use (D.98) and (D.97) and (D.100),

$$z + C_1 = Q_\ell(F_\ell(z)) \quad (\text{D.101})$$

$$F_\ell(z) = Q_\ell^{-1}(z + C_1) \quad (\text{D.102})$$

From (D.100) and Main Paper (14), $Q^{-1}(C_1) = 0$. From (D.100), $C_1 = Q_\ell(0) = 0$. Since $C_1 = 0$ and $F_\ell(z) = Q_\ell^{-1}(z)$, $Q_\ell(q)$ is the quantile function for the random variable z .

To show that there doesn't exist a bounded support solution, assume a $\bar{z} < \infty$. Use Main (15) and (A.27) to get:

$$F_\ell(\bar{z}) = \frac{1}{1+\hat{\lambda}} \quad (\text{D.103})$$

Use the definition of Q_ℓ

$$\bar{z} = Q_\ell^{-1}\left(\frac{1}{1+\hat{\lambda}}\right) \quad (\text{D.104})$$

From (D.100)

$$\bar{z} = \frac{g}{(1+\hat{\lambda})S} {}_1F_2\left(1, 1/\kappa, 1 + 1/\kappa; 1\right) \quad (\text{D.105})$$

But, for any parameters, the Hypergeometric function is only defined on $(-1, 1)$ and goes to infinity on the boundaries, so ${}_1F_2(\cdot; 1)$ diverges. By contradiction, $\bar{z} \rightarrow \infty$ and there can be no stationary equilibrium with bounded support. \square

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