A Appendix: Proofs

A.1 Properties of flow profits and steady-state growth rate

In this appendix section, we prove lemmas 1 and 2.

In the main text, we model market structure as Bertrand competition, generating a sequence of state-dependent flow profits $\{\pi_s, \pi_{-s}\}$ that satisfy properties outlined in Lemma 1, and that $\lim_{s\to\infty}\pi_s = 1$, $\lim_{s\to-\infty}\pi_s = 0$. Our theoretical results hold under any sequence of flow profits that satisfy Lemma 1; hence, our theory nests other market structures. We use $\pi \equiv \lim_{s\to 0}\pi_s$ to denote the limiting total profits in each market, and we exposit using the notation π.

Lemma 1: Follower's flow profits π_{-s} are non-negative, weakly decreasing, and convex; leader's and joint profits (π_s and $(\pi_s + \pi_{-s})$) are bounded, weakly increasing, and eventually concave in s (a sequence $\{a_s\}$ is eventually concave iff there exists $\bar{s}$ such that a_s is concave in s for all $s \geq \bar{s}$).

Proof. Let δ_i be the market share of firm i. The CES demand structure within each market implies that $\delta_i \equiv \frac{p_i y_i}{p_1 y_1 + p_2 y_2} = \frac{p_i^{1-\sigma}}{p_1^{1-\sigma} + p_2^{1-\sigma}}$. Under Bertrand competition, the price charged by a firm with productivity z_i must solve $p_i = \frac{\sigma(1-\delta_i)+\delta_i}{(\sigma-1)(1-\delta_i)}\lambda^{-z_i}$ (recall we normalize wage rate to 1). Aghion et al. (2001) and Atkeson and Burstein (2008) provides detailed derivations of these expressions.

Define ρ_s as an implicit function of the productivity gap: $\rho_s^{\sigma} = \lambda^{-s}\frac{\left(\sigma\rho_s^{\sigma-1}+1\right)}{\sigma+\rho_s^{\sigma-1}}$. It can be verified that flow profits satisfy $\pi_s = \frac{1}{\sigma\rho_s^{\sigma-1}+1}$ for any productivity gap s. The fact that follower's flow profits is convex in s follows from algebra. Moreover, $\lim_{s\to\infty}\rho_s^{\sigma}\lambda^s = 1/\sigma$ and $\lim_{s\to-\infty}\rho_s^{\sigma}\lambda^s = \sigma$; hence, for large s, $\pi_s \approx \frac{1}{\sigma^{\frac{1}{\sigma}}\lambda^{-\frac{\sigma-1}{\sigma}s}+1}$ and $\pi_{-s} \approx \frac{1}{\sigma^{\frac{2\sigma-1}{\sigma}}\lambda^{\frac{\sigma-1}{\sigma}s}+1}$. The eventual concavity of π_s and $(\pi_s + \pi_{-s})$ as $s \to \infty$ is immediate.

Lemma 2: In a steady state, the aggregate productivity growth rate is $g \equiv \ln\lambda\left(\sum_{s=0}^{\infty}\mu_s\eta_s + \mu_0\eta_0\right)$.

Proof The expression $\left(\sum_{s=0}^{\infty} \mu_s \eta_s + \mu_0 \eta_0\right)$ tracks the weighted-average growth rate of the productivity frontier in the economy, i.e., the rate at which markets leave the current state s and move to state $s+1$. In a steady-state, the growth rate of frontier must be the same as the rate at which states fall down by one step, from $s+1$ to s; hence, aggregate growth rate g can also be written as $g = \ln \lambda \left(\sum_{s=1}^{\infty} \mu_s \left(\eta_{-s} + \kappa\right)\right)$.

To prove the expression formally, we proceed in two steps. First, we express aggregate productivity growth as a weighted average of productivity growth in each market. We then use the fact that, given homothetic within-market demand, if a follower in state s improves productivity by one step (i.e. by a factor λ) and a leader in state $s-1$ improves also by one step, the net effect should be equivalent to one step improvement in the overall productivity of a single market.

Aggregate productivity growth is a weighted average of productivity growth in each market:

$$
\begin{aligned}
g = -\frac{d \ln P}{d \ln t} &= -\frac{d \int_0^1 \ln p\left(\nu\right) d\nu}{d \ln t} \\
&= -\sum_{s=0}^{\infty} \mu_s \times \frac{d\left[\int_{z^F} \ln p\left(s, z^F\right) dF\left(z^F\right)\right]}{d \ln t},
\end{aligned}
$$

where we use $\left(s, z^F\right)$ to index for markets in the second line. Now recognize that productivity growth rate in each market, $-\frac{d \ln p\left(s, z^F\right)}{d \ln t}$, is a function of only the productivity gap s and is invariant to the productivity of follower, z^F. Specifically, suppose the follower in market $\left(s, z^F\right)$ experiences an innovation, the market price index becomes $p\left(s-1, z^F+1\right)$. Similarly, if the leader experiences an innovation, the price index becomes $p\left(s+1, z^F\right)$. The corresponding log-changes in price indices are respectively

$$
\begin{aligned}
a_s^F &\equiv \ln p\left(s-1, z^F+1\right) - \ln p\left(s, z^F\right) \\
&= -\ln \lambda + \ln\left[\rho_{s-1}^{1-\sigma} + 1\right]^{\frac{1}{1-\sigma}} - \ln\left[\rho_s^{1-\sigma} + 1\right]^{\frac{1}{1-\sigma}},
\end{aligned}
$$

$$
\begin{aligned}
a_s^L &= \ln p\left(s+1, z^F\right) - \ln p\left(s, z^F\right) \\
&= \ln\left[\rho_{s+1}^{1-\sigma} + 1\right]^{\frac{1}{1-\sigma}} - \ln\left[\rho_s^{1-\sigma} + 1\right]^{\frac{1}{1-\sigma}},
\end{aligned}
$$

where ρ_s is the implicit function defined in the proof for Lemma 1. The log-change in price index is independent of z^F in either case. Hence, over time interval $[t, t+\Delta]$, the change in price index for markets with state variable s at time t follows

$$\Delta \ln p\left(s, z^F\right) = \begin{cases} a_s^L & \text{with probability } \eta_s \Delta, \\ a_s^F & \text{with probability } \left(\eta_{-s} + \kappa \cdot \mathbf{1}\left(s \neq 0\right)\right)\Delta. \end{cases}$$

The aggregate productivity growth can therefore be written as

$$g = -\mu_0 2\eta_0 a_0 - \sum_{s=1}^{\infty} \mu_s \times \left(\eta_s a_s^L + \left(\eta_{-s} + \kappa\right) a_s^F\right).$$

Lastly, note that if both leader and follower in a market experiences productivity improvements, regardless of the order in which these events happen, the price index in the market changes by a factor of λ^{-1}:

$$a_s^F + a_{s-1}^L = a_s^L + a_{s+1}^F = -\ln\lambda \quad \text{for all } s \geq 1.$$

Hence,

$$\begin{aligned} g &= -\mu_0 2\eta_0 a_0 - \sum_{s=1}^{\infty} \mu_s \times \left(\eta_s a_s^L + \left(\eta_{-s} + \kappa\right) a_s^F\right) \\ &= -\mu_0 2\eta_0 a_0 - \sum_{s=1}^{\infty} \mu_s \times \left(\eta_s a_s^L + \left(\eta_{-s} + \kappa\right)\left(-\ln\lambda - a_{s-1}^L\right)\right) \\ &= \ln\lambda \cdot \sum_{s=1}^{\infty} \mu_s \left(\eta_{-s} + \kappa\right) - \left(\sum_{s=1}^{\infty} \mu_s \times \left(\eta_s a_s^L - a_{s-1}^L \left(\eta_{-s} + \kappa\right)\right) + \mu_0 2\eta_0 a_0\right). \end{aligned}$$

Given that steady-state distribution $\{\mu_s\}$ must follow

$$\mu_s\left(\eta_{-s} + \kappa\right) = \begin{cases} \mu_{s-1}\eta_{s-1} & \text{if } s > 1, \\ 2\mu_0\eta_0 & \text{if } s = 1, \end{cases} \tag{10}$$

we know

$$\begin{aligned}
& \sum_{s=1}^{\infty} \mu_s \times \left(\eta_s a_s^L - a_{s-1}^L \left(\eta_{-s} + \kappa \right) \right) + \mu_0 2 \eta_0 a_0 \\
= & \sum_{s=1}^{\infty} \mu_s \eta_s a_s^L + \mu_0 2 \eta_0 a_0 - \left(\sum_{s=1}^{\infty} \mu_s a_{s-1}^L \left(\eta_{-s} + \kappa \right) \right) \\
= & \ 0.
\end{aligned}$$

Hence aggregate growth rate simplifies to $g = \ln \lambda \cdot \sum_{s=1}^{\infty} \mu_s \left(\eta_{-s} + \kappa \right)$, which traces the growth rate of productivity laggards. We can also apply substitutions in (10) again to express productivity growth as a weighted average of frontier growth:

$$g = \ln \lambda \cdot \left(\sum_{s=1}^{\infty} \mu_s \eta_s + 2 \mu_0 \eta_0 \right).$$

A.2 Structure of Equilibrium

It is useful to first understand the structure of value functions given any sequence of (potentially non-equilibrium) investment decisions $\{\eta_s\}_{s=-\infty}^{\infty}$. The fact that firms are forward-looking implies that value function in each state can be written as a weighted average of flow payoffs in all ergodic states induced by the investment decisions, i.e.

$$v_s = \sum_{s'=-\infty}^{\infty} \lambda_{s'|s} \times PV_{s'}, \quad \text{where} \sum_{s'=-\infty}^{\infty} \lambda_{s'|s} = 1 \quad \text{for all } s. \tag{11}$$

The term $PV_{s'} \equiv \frac{\pi_{s'} - c\eta_{s'}}{r}$ represents the permanent value in state s', i.e. the present-discounted value of flow payoff in state s' if the firm stays in that state permanently; $s' > 0$ means the firm is a leader when the productivity gap is s', and $s' < 0$ means the firm is a follower when the productivity gap is $-s'$. In equilibrium, the firm value in state s can be written as a weighted average of the permanent value across all ergodic states. The weight $\lambda_{s'|s}$ can be interpreted as the present-discount fraction of time that the firm is going to be s' steps ahead of his competitor, given that he is currently s steps ahead. The weights $\left\{\lambda_{s'|s}\right\}_{s'=\infty}^{\infty}$ form a measure conditional on the current state s. When the current state s is high, the firm is expected to spend more time in higher indexed states, and the conditional distribution $\left\{\lambda_{s'|s+1}\right\}_{s'=\infty}^{\infty}$ first-order stochastically dominates

$\{\lambda_{s'|s}\}_{s'=\infty}^{\infty}$ for all s.

Likewise, let $w_s \equiv v_s + v_{-s}$ be the joint value of leader and follower in state s. Following the same logic as in equation (11), we can rewrite w_s as a weighted average of the sum of permanent values of leader and follower in every state:

$$w_s = \sum_{s'=0}^{\infty} \tilde{\lambda}_{s'|s} \cdot (PV_{s'} + PV_{-s'}) \,, \quad \text{where } \sum_{s'=0}^{\infty} \tilde{\lambda}_{s'|s} = 1. \tag{12}$$

The weights $\tilde{\lambda}_{s'|s}$ can be interpreted as the present-discounted fraction of time that the state variable is s', i.e. when either firm is s' steps ahead of the other, conditioning on the current gap being s; hence, $\tilde{\lambda}_{s'|s} = \lambda_{s'|s} + \lambda_{s'|-s}$. It is easy to verify that $\{\tilde{\lambda}_{s'|s+1}\}$ first order stochastically dominate $\{\tilde{\lambda}_{s'|s}\}$.

To understand the role of interest rate, note that the firm value in state s can be written as a weighted average of the permanent state payoff in state s and the firm value in neighboring states $s-1$ and $s+1$:

$$v_s = \frac{r}{r+\kappa+\eta_{-s}+\eta_s} \cdot PV_s + \frac{\kappa+\eta_{-s}}{r+\kappa+\eta_{-s}+\eta_s} v_{s-1} + \frac{\eta_s}{r+\kappa+\eta_{-s}+\eta_s} v_{s+1}$$

Holding investment decisions constant, a fall in interest rate r reduces the relative weight on the permanent value of state s, thereby reducing the difference in value functions across states. In fact, holding investment decisions fixed, if there is a state in which the leader chooses not to invest at all ($\eta_{\bar{s}} = 0$ for some $\bar{s}$), then $rv_s \to rv_0$ for all $s \leq \bar{s}$.

We now prove results about the structure of equilibria. For expositional purposes, we assume firms play pure strategies (i.e. they invest at either lower or upper bounds $\eta_s \in \{0, \eta\}$); all of our claims hold for mixed strategy equilibria (i.e. those involving interior investment intensities).

Lemma 3. The leader invests in more states than the follower, $n \geq k$. Moreover, the follower does not invest in states $s = k+1, \cdots, n+1$. Recall $n+1$ is the first state in which market leaders choose not to invest, and $k+1$ is the first state in which followers choose not to invest: $n+1 \equiv \min\{s | s \geq 0, \eta_s < \eta\}$ and $k+1 \equiv \min\{s | s \leq 0, \eta_s < \eta\}$.

Proof Suppose $n < k$, i.e. leader invests in states 1 through n whereas follower invests in states 1 through at least $n+1$. We first show these investment decisions can only be supported by certain lower bounds on the value function of both leader and follower in state $n+1$. We reach for a contradiction, showing that, if $n < k$, then market power is too transient to support these lower bounds on value functions.

The HJB equation for the leader in state $n+2$ implies

$$\begin{aligned} rv_{n+2} &= \max_{\eta_{n+2}\in[0,\eta]} \pi_{n+2} + \eta_{n+2}\left(v_{n+3} - v_{n+2} - c\right) + \left(\eta_{-(n+2)} + \kappa\right)\left(v_{n+1} - v_{n+2}\right) \\ &\geq \pi_{n+2} + \left(\eta + \kappa\right)\left(v_{n+1} - v_{n+2}\right). \end{aligned}$$

The fact that leader does not invest in state $n+1$ implies $c \geq v_{n+2} - v_{n+1}$; combining with the previous inequality, we obtain

$$rv_{n+1} \geq \pi_{n+2} - c\left(\eta + \kappa + r\right).$$

The HJB equation for the follower in state $n+1$ implies

$$\begin{aligned} rv_{-(n+1)} &= \max_{\eta_{-(n+1)}\in[0,\eta]} \pi_{-(n+1)} + \left(\eta_{-(n+1)} + \kappa\right)\left(v_{-n} - v_{-(n+1)}\right) - c\eta_{-(n+1)} \\ &\geq \pi_{-(n+1)} + \kappa\left(v_{-n} - v_{-(n+1)}\right). \end{aligned}$$

The fact that follower chooses to invest in state $n+1$ implies $c \leq v_{-n} - v_{-(n+1)}$; combining with the previous inequality, we obtain

$$rv_{-(n+1)} \geq \pi_{-(n+1)} + c\kappa. \tag{13}$$

Combining this with the earlier inequality involving rv_{n+1}, we obtain an inequality on the joint value in state $n+1$:

$$rw_{n+1} \geq \pi_{n+2} + \pi_{-(n+1)} - c\left(\eta + r\right) \tag{14}$$

We now show that inequalities (13) and (14) cannot both be true. To do so, we construct alternative economic environments with value functions that dominate w_{n+1} and $v_{-(n+1)}$; we then show that even these dominating value functions cannot satisfy both inequalities.

First, fix n and fix investment strategies (leader invests until state $n+1$ and follower invests at least through $n+1$); suppose for all states $1 \leq s \leq n+1$, follower's profits are equal to $\pi_{-(n+1)}$ and leader's profits are equal to π_{n+2}; two firms each earn $\frac{\pi_{-(n+1)}+\pi_{n+2}}{2}$ in state zero. The joint profits in this modified economic environment are independent of the state by construction; moreover, the joint flow profits always weakly dominate those in the original environment and strictly dominate in state zero ($\pi_{n+2} + \pi_{-(n+1)} \geq \pi_1 + \pi_{-1} > 2\pi_0$). Let $\hat{w}_s$ denote the value function in the modified environment; $\hat{w}_s > w_s$ for all $s \leq n+1$.

Consider the joint value in this modified environment but under alternative investment strategies. Let $\bar{n}$ index for investment strategies: leader invests in states 1 through $\bar{n}$ whereas the follower invests at least through $\bar{n}+1$. Let $\hat{w}_s^{(\bar{n})}$ denote the joint value in state s under investments indexed by $\bar{n}$; we argue that $\hat{w}_{\bar{n}+1}^{(\bar{n})}$ is decreasing in $\bar{n}$. To see this, note that the joint flow payoffs in all states 0 through $\bar{n}$ is constant by construction and is equal to $\left(\pi_{n+2} + \pi_{-(n+1)} - 2c\eta\right)$—total profits net of investment costs. The joint flow payoff in state $\bar{n}+1$ is $\left(\pi_{n+2} + \pi_{-(n+1)} - c\eta\right)$. Hence, the joint market value in state $\bar{n}+1$ under the investment strategies indexed by $\bar{n}$ is equal to

$$\hat{w}_{\bar{n}+1}^{(\bar{n})} = \frac{\pi_{n+2} + \pi_{-(n+1)} - 2c\eta\left(1 - \tilde{\lambda}_{\bar{n}+1|\bar{n}+1}^{(\bar{n})}/2\right)}{r},$$

where $\tilde{\lambda}_{\bar{n}+1|\bar{n}+1}^{(\bar{n})}$ is the present discount fraction of time that the market spends in state $\bar{n}+1$, conditioning on the current state is $\bar{n}+1$, and that firms follow investment strategies indexed by $\bar{n}$. The object $\tilde{\lambda}_{\bar{n}+1|\bar{n}+1}^{(\bar{n})}$ is decreasing in $\bar{n}$: the more states in which both firms invest, the less time that the market will spend in the state $\bar{n}+1$ in which only one firm (the follower) invests. Hence, $\hat{w}_{\bar{n}+1}^{(\bar{n})}$ is decreasing in $\bar{n}$, and that $\hat{w}_1^{(0)} \geq \hat{w}_{n+1}^{(n)} > w_{n+1}$. The same logic also implies $\hat{v}_0^{(0)} = \frac{1}{2}\hat{w}_0^{(0)} > \frac{1}{2}w_0 = v_0$.

The follower's value $\hat{v}_{-1}^{(0)}$, in the alternative environment, when investment strategies are indexed by zero (i.e. firms invest in states 0 and -1 only), is higher than $v_{-(n+1)}$. This is because

$$\begin{aligned}\hat{v}_{-1}^{(0)} &= \frac{\pi_{-(n+1)} - c\eta + \kappa\hat{v}_0^{(0)}}{r+\kappa+\eta} \\ &> \frac{\pi_{-(n+1)} - c\eta + \kappa v_0}{r+\kappa+\eta} \\ &\geq \frac{\pi_{-(n+1)} - c\eta + \kappa v_{-n}}{r+\kappa+\eta} \\ &= v_{-(n+1)}.\end{aligned}$$

We now show that the inequalities $r\hat{v}_{-1}^{(0)} \geq \pi_{-(n+1)} + c\kappa$ and $r\hat{w}_1^{(0)} \geq \pi_{n+2} + \pi_{-(n+1)} - c(\eta + r)$ cannot both hold. We can explicitly solve for the value functions from the HJB equations:

$$\hat{w}_0^{(0)} = \frac{\pi_{n+2} + \pi_{-(n+1)} - 2c\eta + 2\eta\hat{w}_1^{(0)}}{r+2\eta}$$

$$\hat{w}_1^{(0)} = \frac{\pi_{n+2} + \pi_{-(n+1)} - c\eta + (\eta+\kappa)\hat{w}_0^{(0)}}{r+\eta+\kappa}$$

$$\hat{v}_{-1}^{(0)} = \frac{\pi_{-(n+1)} - c\eta + (\eta+\kappa)\hat{w}_0^{(0)}/2}{r+\eta+\kappa}$$

Solving for $\hat{w}_1^{(0)}$ and $\hat{v}_{-1}^{(0)}$, we obtain

$$r\hat{w}_1^{(0)} = \pi_{n+2} + \pi_{-(n+1)} - c\eta\left(1 + \frac{\eta+\kappa}{r+3\eta+\kappa}\right)$$

$$(r+\eta+\kappa)\, r\hat{v}_{-1}^{(0)} = r\left(\pi_{-(n+1)} - c\eta\right) + (\eta+\kappa)\left(\frac{\pi_{n+2} + \pi_{-(n+1)}}{2} - c\eta\frac{r+2\eta+\kappa}{r+3\eta+\kappa}\right)$$

That $r\hat{v}_{-1}^{(0)} \geq \pi_{-(n+1)} + c\kappa$ implies

$$\begin{aligned}&(r+\eta+\kappa)\, r\hat{v}_{-1}^{(0)} \\ =\; & r\left(\pi_{-(n+1)} - c\eta\right) + (\eta+\kappa)\left(\frac{\pi_{n+2} + \pi_{-(n+1)}}{2} - c\eta\frac{r+2\eta+\kappa}{r+3\eta+\kappa}\right) \\ \geq\; & (r+\eta+\kappa)\left(\pi_{-(n+1)} + c\kappa\right)\end{aligned}$$

$$\Longrightarrow (\eta+\kappa)\left(\frac{\pi_{n+2} - \pi_{-(n+1)}}{2} - c\eta\frac{r+2\eta+\kappa}{r+3\eta+\kappa}\right) \geq (r+\eta+\kappa)\, c\kappa + c\eta r$$

Since $\frac{\pi_{n+2}-\pi_{-(n+1)}}{2} \leq \frac{\pi_{n+2}}{2} < c\eta$, it must be the case that

$$(\eta+\kappa)\,c\eta > (r+\eta+\kappa)\,c\kappa + c\eta r + (\eta+\kappa)\,c\eta\frac{r+2\eta+\kappa}{r+3\eta+\kappa}.$$

On the other hand, that $r\hat{w}_1^{(0)} \geq \pi_{n+2} + \pi_{-(n+1)} - c\,(\eta+r)$ implies $r \geq \eta\frac{\eta+\kappa}{r+3\eta+\kappa}$; hence the previous inequality implies

$$\begin{aligned}(\eta+\kappa)\,c\eta &> (r+\eta+\kappa)\,c\kappa + (\eta+\kappa)\,c\eta\frac{\eta}{r+3\eta+\kappa} + (\eta+\kappa)\,c\eta\frac{r+2\eta+\kappa}{r+3\eta+\kappa} \\ &= (r+\eta+\kappa)\,c\kappa + (\eta+\kappa)\,c\eta,\end{aligned}$$

which is impossible; hence $n \geq k$.

We now show that the follower does not invest in states $s \in \{k+1, ..., n+1\}$. First, note

$$\begin{aligned}(r+\eta+\kappa)\,(v_{-s}-v_{-s-1}) &= \pi_{-s}-\pi_{-s-1}+\kappa\,(v_{-s+1}-v_s)+\eta\,(v_{-s-1}-v_{-s-2}) \\ &\quad +\max\{\eta\,(v_{-s+1}-v_{-s}-c)\,,0\}-\max\{\eta\,(v_{-s}-v_{-s-1}-c)\,,0\}\,.\end{aligned}$$

Suppose $v_{-s+1}-v_{-s} \geq (v_{-s}-v_{-s-1})$, then

$$(r+\eta+\kappa)\,(v_{-s}-v_{-s-1}) \geq \pi_{-s}-\pi_{-s-1}+\kappa\,(v_{-s+1}-v_{-s})+\eta\,(v_{-s-1}-v_{-s-2})$$

$$\Longrightarrow\ (r+\eta)\,(v_{-s}-v_{-s-1}) \geq \pi_{-s}-\pi_{-s-1}+\eta\,(v_{-s-1}-v_{-s-2})\,. \tag{15}$$

If $v_{-s+1}-v_{-s} < (v_{-s}-v_{-s-1})$, then

$$\begin{aligned}(r+\eta)\,(v_{-s}-v_{-s-1}) &< \pi_{-s}-\pi_{-s-1}+\eta\,(v_{-s-1}-v_{-s-2}) \\ &\quad +\max\{\eta\,(v_{-s+1}-v_{-s}-c)\,,0\} \\ &\quad -\max\{\eta\,(v_{-s}-v_{-s-1}-c)\,,0\} \\ &\leq \pi_{-s}-\pi_{-s-1}+\eta\,(v_{-s-1}-v_{-s-2})\,. \end{aligned} \tag{16}$$

Now suppose $\eta_{-k-1} = 0$ but $\eta_{-s'} = \eta$ for some $s' \in \{k+2, ..., n+1\}$. This implies

$$v_{-(k-1)} - v_{-k} \geq c > v_{-k} - v_{-k-1} < v_{-s'+1} - v_{-s'},$$

implying there must be at least one $s \in \{k+2, ..., n+1\}$ such that $v_{-s+1} - v_{-s} \geq v_{-s} - v_{-s-1} < v_{-s-1} - v_{s-2}$. Inequalities (15) and (16) implies

$$(r+\eta)(v_{-s} - v_{-s-1}) \geq \pi_{-s} - \pi_{-s-1} + \eta(v_{-s-1} - v_{-s-2}) \tag{17}$$

$$(r+\eta)(v_{-s-1} - v_{-s-2}) < \pi_{-s-1} - \pi_{-s-2} + \eta(v_{-s-2} - v_{-s-3})$$

Inequality (17) and $v_{-s} - v_{-s-1} < v_{-s-1} - v_{-s-2}$ implies $r(v_{-s} - v_{-s-1}) > \pi_{-s} - \pi_{-s-1}$; convexity in follower's profit functions further implies $r(v_{-s} - v_{-s-1}) > \pi_{-s-1} - \pi_{-s-2}$. Hence it must be the case that $(v_{-s-2} - v_{s-3}) > (v_{-s-1} - v_{-s-2})$. Applying inequality (16) again,

$$(r+\eta)(v_{-s-2} - v_{-s-3}) < \pi_{-s-2} - \pi_{-s-3} + \eta(v_{-s-3} - v_{-s-4}).$$

That $r(v_{-s-2} - v_{-s-3}) > \pi_{-s-2} - \pi_{-s-3}$ further implies $(v_{-s-3} - v_{-s-4}) > (v_{-s-2} - v_{-s-3})$. By induction, we can show

$$v_{s-1} - v_{s-2} < v_{s-2} - v_{s-3} < \cdots < v_{-n} - v_{-(n+1)}.$$

But

$$(r+\eta+\kappa)\left(v_{-n} - v_{-(n+1)}\right) \leq \pi_{-n} - \pi_{-(n+1)} + \kappa(v_{s+1} - v_s) + \eta(v_{-n+1} - v_{-n+1})$$

$$\implies (r+\eta)\left(v_{-n} - v_{-(n+1)}\right) \leq \pi_{-n} - \pi_{-(n+1)}$$

which is a contradiction, given convexity of the profit functions. Hence, we have shown $v_{-k} - v_{-(k+1)} \geq v_{-s} - v_{-s-1}$ for all $s \in \{k+1, ..., n+1\}$, establishing that follower cannot invest in these states.

Lemma 4: In a steady-state induced by investment cutoffs (n,k), the aggregate productivity growth rate is $g = \ln\lambda \cdot \left(\mu^C(\eta+\kappa) + \mu^M\kappa\right)$, where μ^C is the fraction of markets in the competitive region ($\mu^C = \sum_{s=1}^{k}\mu_s$) and μ^M is the fraction of markets in the monopolistic region ($\mu^M = \sum_{s=k+1}^{n+1}\mu_s$). The fraction of markets in each region satisfies

$$\mu_0 + \mu^C + \mu^M = 1, \quad \mu_0 \propto (\kappa/\eta)^{n-k+1}(1+\kappa/\eta)^k,$$

$$\mu^C \propto (\kappa/\eta)^{n-k}\left((1+\kappa/\eta)^k - 1\right), \quad \mu^M \propto \frac{1-(\kappa/\eta)^{n-k+1}}{1-\kappa/\eta}.$$

Proof. Given the cutoff strategies (n,k), aggregate productivity growth is (from Lemma 3)

$$g = \ln\lambda \cdot \left(\sum_{s=1}^{n}\mu_s\eta + 2\mu_0\eta\right).$$

The steady-state distribution must follow

$$\mu_s\eta = \begin{cases} \mu_1(\eta+\kappa)/2 & \text{if } s=0 \\ \mu_{s+1}(\eta+\kappa) & \text{if } 1\le s\le k-1 \\ \mu_{s+1}\kappa & \text{if } k\le s\le n+1 \\ 0 & \text{if } s>n+1 \end{cases}$$

Hence we can rewrite the aggregate growth rate as

$$\begin{aligned} g &= \ln\lambda \cdot \left(2\mu_0\eta + \sum_{s=1}^{k-1}\mu_s\eta + \sum_{s=k-1}^{n}\mu_s\eta\right) \\ &= \ln\lambda \cdot \left(\mu_1(\eta+\kappa) + \sum_{s=2}^{k}\mu_s(\eta+\kappa) + \sum_{s=k}^{n+1}\mu_s\kappa\right) \\ &= \ln\lambda \cdot \left(\mu^C(\eta+\kappa) + \mu^M\kappa\right), \end{aligned}$$

as desired.

To solve for μ_0, μ^C, and μ^M as functions of n and k, note that steady-state distribution follows:

$$\mu_s \eta = \begin{cases} \mu_1 \left(\eta + \kappa\right)/2 & \text{if } s = 0 \\ \mu_{s+1} \left(\eta + \kappa\right) & \text{if } 1 \leq s \leq k-1 \\ \mu_{s+1} \kappa & \text{if } k \leq s \leq n+1 \\ 0 & \text{if } s > n+1. \end{cases}$$

We can rewrite μ_s as a function of μ_{n+1} for all s. Let $\alpha \equiv \kappa/\eta$, then

$$\mu_s = \begin{cases} \mu_{n+1} \alpha^{n+1-s} & \text{if } n+1 \geq s \geq k \\ \mu_{n+1} \alpha^{n+1-k} \left(1+\alpha\right)^{k-s} & \text{if } k-1 \geq s \geq 0 \end{cases}$$

Hence $\mu_0 = \mu_{n+1} \alpha^{n+1-k} \left(1+\alpha\right)^k$. The fraction of markets in the competitive and monopolistic regions can be written, respectively, as

$$\mu^M = \mu_{n+1} \sum_{s=k+1}^{n+1} \alpha^{n+1-s} = \mu_{n+1} \frac{1-\alpha^{n-k+1}}{1-\alpha}$$

$$\mu^C = \mu_{n+1} \alpha^{n+1-k} \sum_{s=1}^{k} \left(1+\alpha\right)^{k-s} = \mu_{n+1} \alpha^{n-k} \left(\left(1+\alpha\right)^k - 1 \right).$$

Lemma 5: If follower invests in state 1, then the steady-state aggregate productivity growth is bounded below by $\ln \lambda \cdot \kappa$, the step-size of productivity increments times the rate of technology diffusion.

Proof. Given $k \geq 1$, the fraction of markets in the competitive region can be written as

$$
\begin{aligned}
\mu^C &= \sum_{s=1}^{k} \mu_s \\
&= \mu_1 + \underbrace{\mu_1 (1+\alpha)^{-1}}_{=\mu_2} + \cdots + \underbrace{\mu_1 (1+\alpha)^{-(k-1)}}_{=\mu_k} \\
&= \underbrace{\mu_0 \frac{\kappa+\eta}{2\eta}}_{=\mu_1} \frac{1-(1+\alpha)^{-k}}{1-(1+\alpha)^{-1}} \\
&\geq \mu_0 \frac{\kappa+\eta}{2\eta}
\end{aligned}
$$

Aggregate growth rate can be re-written as

$$
\begin{aligned}
g &= \ln \lambda \cdot \left[(1-\mu_0)\,\kappa + \mu^C \eta \right] \\
&\geq \ln \lambda \cdot \left[(1-\mu_0)\,\kappa + \mu_0 \frac{\kappa+\eta}{2} \right] \\
&\geq \ln \lambda \cdot \left[(1-\mu_0)\,\kappa + \mu_0 \kappa \right] \\
&= \ln \lambda \cdot \kappa,
\end{aligned}
$$

as desired.

A.3 Asymptotic Results as $r \to 0$

Lemma A.1. $\Delta w_0 \equiv w_1 - w_0 = \frac{rw_0+2c\eta-2\pi_0}{2\eta}$**;** Δw_0 **is bounded away from zero.**

Proof The equality from the HJB equation $rw_0 = 2\pi_0 - 2c\eta + 2\eta\,(w_1 - w_0)$. That Δw_0 is bounded away from zero follows from the fact that $rw_0 \geq 0$ and assumption 1 ($2c\eta > \pi \equiv \lim_{s\to 0} \pi_s + \pi_{-s} > 2\pi_0$). QED.

A.3.1 Mathematical Preliminaries

Consider the following recursive formulation of value functions:

$$ru_{s+1} = \lambda + p\left(u_s - u_{s+1}\right) + q\left(u_{s+2} - u_{s+1}\right)$$

The HJB equation states that, starting from state s, there's a Poisson rate p of moving up one state, and rate q of moving down; the flow payoff is λ and discount rate is r.

Fix a state s. Given u_s and $\Delta u_s \equiv u_{s+1} - u_s$, we can solve for all u_{s+t} with $t > 0$ as recursive functions of u_s and Δu_s; for instance,

$$u_{s+2} - u_{s+1} = \frac{ru_s - \lambda}{q} + \left(\frac{p+r}{q}\right)\Delta u_s,$$

$$u_{s+3} - u_{s+3} = \frac{ru_s - \lambda}{q} + \left(\frac{p+r}{q}\right)\left(u_{s+2} - u_{s+1}\right) + \frac{r\Delta u_s}{q},$$

and so on. The recursive formulation generically does not have a nice closed-form representation, as the number of terms quickly explodes as we expand out the recursion. However, as $r \to 0$, the value functions do emit asymptotic closed form expressions, as Proposition A.1 shows. In what follows, let $\sim$ denote asymptotic equivalence as $r \to 0$, i.e. $x \sim y$ iff $\lim_{r\to 0}\left(x - y\right) = 0$.

Proposition A.1. Let $\delta \equiv \frac{ru_s - \lambda}{q}$, $a \equiv p/q$, $b \equiv r/q$, then for all $t > 0$,

$$\begin{aligned} u_{s+t} - u_s \sim\; & \left(\Delta u_s\right)\frac{1-a^t}{1-a} + \delta\frac{t - \frac{a-a^t}{1-a}}{1-a} \\ & + \Delta u_s \cdot b\frac{\left(t-1\right)\left(1+a^t\right)\left(1-a\right) - \left(2-a\right)\left(a^t - a\right)}{\left(1-a\right)^3} \\ & + \delta b\frac{1}{\left(1-a\right)^3}\left(\frac{\left(t-2\right)\left(t-1\right)}{2}\left(1-a\right) - \left(t-3\right)a^t - a\left(2-a\right)\left(t-1\right) + 2a\left(1-a\right)\right) \end{aligned}$$

$$
\begin{aligned}
u_{s+t}-u_{s+t-1} \sim\; & \Delta u_s a^{t-1}+\delta\frac{1-a^{t-1}}{1-a} \\
& +\Delta u_s b\frac{\left(\left(t-1\right)\left(1+a^{t}\right)-\left(t-2\right)\left(1+a^{t-1}\right)\right)\left(1-a\right)-\left(2-a\right)\left(a^{t}-a^{t-1}\right)}{\left(1-a\right)^{3}} \\
& +\delta b\frac{1}{\left(1-a\right)^{3}}\left(\frac{\left(t-2\right)\left(t-1\right)-\left(t-2\right)\left(t-3\right)}{2}\left(1-a\right)-\left(t-3\right)a^{t}+\left(t-4\right)a^{t-1}-a\left(2-a\right)\right)
\end{aligned}
$$

The following simplifications of the formulas will be useful if $\lim_{r\to 0} t \to \infty$:

1. when $a<1$:

$$u_{s+t}-u_{s+t-1}\sim \Delta u_s a^{t-1}+\frac{\delta}{1-a}+\frac{b\Delta u_s}{\left(1-a\right)^{2}};$$

   (a) if $r\Delta u_s \to 0$,

$$u_{s+t}-u_s \sim \Delta u_s\frac{1}{1-a}+\frac{t\delta}{1-a};$$

   (b) if $r\Delta u_s \not\to 0$,

$$r\left(u_{s+t}-u_s\right)\sim \frac{r\Delta u_s}{1-a}.$$

2. when $a>1$, $r\Delta u_s\to 0$, and $\Delta u_s+\frac{\delta}{a-1}\not\sim 0$,

$$r\left(u_{s+t}-u_s\right)\sim\left(\Delta u_s+\frac{\delta}{a-1}\right)\frac{ra^{t}}{a-1},$$

$$r\left(u_{s+t}-u_{s+t-1}\right)\sim\left(\Delta u_s+\frac{\delta}{a-1}\right)ra^{t-1}.$$

   If $\Delta u_s+\frac{\delta}{a-1}\sim 0$,

$$u_{s+t}-u_s\sim -\frac{b\delta}{\left(1-a\right)^{4}}\cdot a^{t+1}.$$

Suppose the flow payoffs are state-dependent $\{\lambda_s\}$, i.e.

$$ru_{s+1}=\lambda_{s+1}+p\left(u_s-u_{s+1}\right)+q\left(u_{s+2}-u_{s+1}\right)$$

If λ is an upper bound for $\{\lambda_s\}$, then the formulas provide asymptotic lower bounds for $u_{s+t}-u_{s+t-1}$ and u_{s+t} as functions of u_s and Δu_s. Conversely, if λ is a lower bound for $\{\lambda_s\}$, then the formulas provide asymptotic upper bounds for $u_{s+t} - u_{s+t-1}$ and u_{s+t}.

One can analogously write u_s and Δu_s as asymptotic functions of Δu_{s+t} and u_{s+t}.

Proof of Proposition A.1. The recursive formulation can be re-written as

$$u_{s+1} - u_s = \Delta u_s$$

$$\begin{aligned} u_{s+2} - u_{s+1} &= a\left(u_{s+1} - u_s\right) + \frac{r\left(u_{s+1} - u_s\right) + ru_s - v}{q} \\ &= a\Delta u_s + b\Delta u_s + \delta \end{aligned}$$

$$u_{s+2} - u_s = (1+a)\,\Delta u_s + b\Delta u_s + \delta$$

Likewise,

$$u_{s+3} - u_{s+2} = a^2\Delta u_s + (1+2a)\,b\Delta u_s + (1+a)\,\delta + o\left(r^2\right)$$

$$u_{s+3} - u_s = \left(1 + a + a^2\right)\Delta u_s + (1+1+2a)\,b\Delta u_s + (1+1+a)\,\delta + b\delta + o\left(r^2\right)$$

Applying the formula iteratively, one can show that

$$u_{s+t+1} - u_{s+t} = a^t\Delta u_s + \delta\sum_{z=0}^{t-1} a^z + b\Delta u_s\sum_{z=1}^{t} za^{z-1} + b\delta\sum_{z=1}^{t-1}\sum_{m=1}^{z} ma^{m-1} + o\left(r^2\right)$$

$$u_{s+t+1} - u_s = \Delta u_s\sum_{z=0}^{t} a^z + \delta\sum_{z=0}^{t}\sum_{m=0}^{z-1} a^m + b\Delta u_s\sum_{z=1}^{t}\sum_{m=1}^{z} ma^{m-1} + b\delta\sum_{x=1}^{t-1}\sum_{z=1}^{x}\sum_{m=1}^{z} ma^{m-1} + o\left(r^2\right)$$

One obtains the Lemma by applying the following formulas for power series summation:

$$\sum_{z=0}^{t} a^z = \frac{1 - a^{t+1}}{1 - a}$$

$$\sum_{z=0}^{t}\sum_{m=0}^{z-1} a^m = \frac{t + 1 - \frac{a - a^{t+1}}{1-a}}{1 - a}$$

$$\sum_{z=1}^{t}\sum_{m=1}^{z} ma^{m-1} = \frac{t\left(1+a^{t+1}\right)(1-a)-(2-a)\left(a^{t+1}-a\right)}{(1-a)^{3}}$$

$$\sum_{x=1}^{t-1}\sum_{z=1}^{x}\sum_{m=1}^{z} ma^{m-1} = \frac{1}{(1-a)^{3}}\left(\frac{t\,(t-1)}{2}(1-a)-(t-2)\,a^{t+1}-a\,(2-a)\,t+2a\,(1-a)\right).$$

The third and fourth summations formulas follow because

$$\begin{aligned}
\sum_{m=1}^{z} ma^{m-1} &= \left(1+2a+3a^{2}+\cdots+za^{z-1}\right)\\
&= \left(\frac{1-a^{z}}{1-a}+a\frac{1-a^{z-1}}{1-a}+\cdots+a^{z-1}\frac{1-a}{1-a}\right)\\
&= \left(\frac{1+a+\cdots+a^{z-1}}{1-a}-\frac{za^{z}}{1-a}\right)\\
&= \left(\frac{1-a^{z}}{(1-a)^{2}}-\frac{za^{z}}{1-a}\right)
\end{aligned}$$

$$\begin{aligned}
\sum_{z=1}^{s}\sum_{m=1}^{z} ma^{m-1} &= \sum_{z=1}^{s}\left(\frac{1-a^{z}}{(1-a)^{2}}-\frac{za^{z}}{1-a}\right)\\
&= \frac{s}{(1-a)^{2}}-\frac{a-a^{s+1}}{(1-a)^{2}}-\frac{a}{1-a}\sum_{z=1}^{s} za^{z-1}\\
&= \frac{s}{(1-a)^{2}}-\frac{a-a^{s+1}}{(1-a)^{2}}-\frac{a}{1-a}\left(\frac{1-a^{s}}{(1-a)^{2}}-\frac{sa^{s}}{1-a}\right)\\
&= \frac{s\,(1-a)}{(1-a)^{3}}-\frac{a\,(1-a)-(1-a)\,a^{s+1}}{(1-a)^{3}}-\frac{a-a^{s+1}}{(1-a)^{3}}+\frac{sa^{s+1}\,(1-a)}{(1-a)^{3}}\\
&= \frac{s\left(1+a^{s+1}\right)(1-a)-(2-a)\left(a^{s+1}-a\right)}{(1-a)^{3}}
\end{aligned}$$

$$\begin{aligned}
\sum_{x=1}^{s-1}\sum_{z=1}^{x}\sum_{m=1}^{z} ma^{m-1} &= \sum_{x=1}^{s-1}\frac{x\left(1+a^{x+1}\right)(1-a)-(2-a)\left(a^{x+1}-a\right)}{(1-a)^3} \\
&= \frac{1}{(1-a)^3}\left(\sum_{x=1}^{s-1} x(1-a)+xa^{x+1}(1-a)-(2-a)\left(a^{x+1}-a\right)\right) \\
&= \frac{1}{(1-a)^3}\left(\frac{s(s-1)}{2}(1-a)+a^2(1-a)\underbrace{\sum_{x=1}^{s-1} xa^{x-1}}_{=\frac{1-a^{s-1}}{(1-a)^2}-\frac{(s-1)a^{s-1}}{1-a}}\right. \\
&\qquad \left. -a(2-a)(s-1)-(2-a)a^2\frac{1-a^{s-1}}{1-a}\right) \\
&= \frac{1}{(1-a)^3}\left(\frac{s(s-1)}{2}(1-a)-(s-2)a^{s+1}-a(2-a)s+2a(1-a)\right)
\end{aligned}$$

A.3.2 Proofs of Lemma 6: $\lim_{r\to 0} k = \lim_{r\to 0}(n-k) = \infty$

Recall n and k are the last states in which the leader and the follower, respectively, chooses to invest in an equilibrium. Both n and k are functions of the interest rate r. Also recall that we use $w_s \equiv v_s + v_{-s}$ to denote the total firm value of a market in state s.

We first prove $\lim_{r\to 0}(n-k) = \infty$**.**

Suppose k and $(n-k)$ are both bounded as $r \to 0$; let N be an upper bound for n, i.e. $N \geq n(r)$ for all r.

Consider the sequence of value functions $\hat{v}_s$ under alternative investment decisions: leader follows equilibrium strategies and invests in $n(r)$ states whereas follower does not invest at all. The sequence of value function dominates the equilibrium value functions ($\hat{v}_s \geq v_s$) for all $s \geq 0$, because:

1. The joint value is higher in every state $\hat{w}_s \geq w_s$, because flow payoffs are weakly higher and that the value functions are placing higher weights on higher states (which have higher flow payoffs). Hence the firm value in state zero is higher $\hat{v}_0 \geq v_0$.
2. The leader's value function can be written as a weighted average of flow payoffs in $s > 0$ and the value of being in state zero; the flow payoffs are the same for all $s > 0$, and $\hat{v}_0 \geq v_0$.

Furthermore when follower does not invest, the leader's value function always places higher weights in states with higher payoffs; hence $\hat{v}_s \geq v_s$ for all $s > 0$.

We now look for a contradiction. As $r \to 0$,

$$r\hat{v}_{N+1} = \frac{r\pi_{N+1} + \kappa r\hat{v}_N}{r+\kappa} \to r\hat{v}_N,$$

$$r\hat{v}_N = \frac{r(\pi_N - c\eta_N) + \kappa r\hat{v}_{N-1} + \eta_N r\hat{v}_{N+1}}{r+\kappa+\eta_N} \to r\hat{v}_{N-1},$$

and so on. By induction, $r\hat{v}_s \sim r\hat{v}_0$ for all $-N+1 \leq s \leq N+1$.

Also note that leader stops investing in state $n+1$ implies

$$\lim_{r\to 0} rv_{n+1} \geq \lim_{r\to 0} \pi_{n+2} - c\kappa,$$

thus $\lim_{r\to 0} r\hat{v}_0 \geq \lim_{r\to 0} \pi_{n+2} - c\kappa$.

Lastly, note $\Delta\hat{v}_0 \geq \Delta\hat{w}_0 = \frac{r\hat{w}_0 - (2\pi_0 - 2c\eta)}{2\eta} = \frac{r\hat{v}_0 - (\pi_0 - c\eta)}{\eta}$.

Putting these pieces together, we apply Proposition A.1 to compute a lower bound for $\Delta\hat{v}_n$ as a function of $\hat{v}_0$ and $\Delta\hat{v}_0$ (substituting $u_s = \hat{v}_0$, $u_{s+t} = \hat{v}_{n+1}$, $a = \kappa/\eta$, $b = r/\eta$, $\delta = \frac{r\hat{v}_0 - (\pi_{n+2} - c\eta)}{\eta}$):

$$\begin{aligned}
\lim_{r\to 0} \Delta\hat{v}_{n+1} &\geq \lim_{r\to 0}\left(\Delta\hat{v}_0 (\kappa/\eta)^n + \frac{r\hat{v}_0 - (\pi_{n+2} - c\eta)}{\eta}\frac{1-(\kappa/\eta)^n}{1-\kappa/\eta}\right)\\
&\geq \lim_{r\to 0} \frac{r\hat{v}_0 - (\pi_0 - c\eta)}{\eta}(\kappa/\eta)^n + \frac{r\hat{v}_0 - (\pi_{n+2} - c\eta)}{\eta}\frac{1-(\kappa/\eta)^n}{1-\kappa/\eta}\\
&\geq \lim_{r\to 0} \frac{\pi_{n+2} - c\kappa - (\pi_0 - c\eta)}{\eta}(\kappa/\eta)^n + \frac{\pi_{n+2} - c\kappa - (\pi_{n+2} - c\eta)}{\eta}\frac{1-(\kappa/\eta)^n}{1-\kappa/\eta}\\
&> \lim_{r\to 0} c(\kappa/\eta)^n + \frac{c(\eta-\kappa)}{\eta}\frac{1-(\kappa/\eta)^n}{1-\kappa/\eta}\\
&= c,
\end{aligned}$$

where the last inequality follows from assumption 1, that $\pi_{n+2} - \pi_0 \geq \pi_1 - \pi_0 > c\kappa$. But this is a contradiction to the claim that leader stops investing in state $n+1$ (i.e. $\Delta\hat{v}_{n+1} \leq c$ for any r).

Next, suppose $\lim_{r\to 0} k = \infty$ but $(n-k)$ remain bounded. Let $\epsilon \equiv 2c\eta - \lim_{s\to\infty}(\pi_s + \pi_{-s})$; $\epsilon > 0$ under assumption 1. The joint flow payoff $\pi_s + \pi_{-s} - 2c\eta$ is negative and bounded above by $-\epsilon$ in all states $s \leq k$. The joint market value in state 0 is

$$\begin{aligned} w_0 &= \sum_{s'=0}^{k} \tilde{\lambda}_{s'|0} \cdot (PV_{s'} + PV_{-s'}) + \sum_{s'=k+1}^{n+1} \tilde{\lambda}_{s'|0} \cdot (PV_{s'} + PV_{-s'}) \\ &\leq \frac{-\epsilon}{r} \cdot \left(\sum_{s'=0}^{k} \tilde{\lambda}_{s'|0} \right) + \sum_{s'=k+1}^{n+1} \tilde{\lambda}_{s'|0} \cdot (PV_{s'} + PV_{-s'}) \,. \end{aligned}$$

As $k \to \infty$ while $n-k$ remain bounded, the present-discount fraction of time that the market spends in states $s \leq k$ converges to 1 ($\sum_{s'=0}^{k} \tilde{\lambda}_{s'|0} \to 1$), implying that $\lim_{r\to 0} rw_0$ is negative. Since firms can always ensure non-negative payoffs by not taking any investment, this cannot be an equilibrium, reaching a contradiction. Hence $\lim_{r\to 0} (n-k) = \infty$.

To show $\lim_{r\to 0} k = \infty$, we first establish a few additional asymptotic properties of the model.

Lemma A.2. The following statements are true:

1. $rv_n \sim \pi - c\kappa$, where $\pi \equiv \lim_{s\to\infty} \pi_s$.

2. $v_{n+1} - v_n \sim c$.

3. $r(n-k) \sim 0$.

4. $rk \sim 0$.

Proof

1. The claim follows from the fact that if firm invests in state n but not in state $n+1$, then

$$v_{n+2} - v_{n+1} = \frac{\pi_{n+2} - rv_{n+1}}{r+\kappa} \leq c$$

$$v_{n+1} - v_n = \frac{\pi_{n+1} - rv_n}{r+\kappa} \geq c$$

implying

$$\pi - c\kappa = \lim_{r\to 0} (\pi_{n+2} - c\kappa) \geq \lim_{r\to 0} rv_n \geq \lim_{r\to 0} (\pi_{n+1} - c\kappa) = \pi - c\kappa,$$

as desired.

2. The claim follows from the previous one: $v_{n+1} - v_n = \frac{\pi_{n+1}}{r+\kappa} - \frac{rv_n}{r+\kappa} \sim \frac{\pi - rv_n}{\kappa} \sim c$.

3. The previous claims show $rv_n \sim \pi - c\kappa$ and $\Delta v_n \sim c$. We apply Proposition A.1 to iterate backwards and obtain

$$\begin{aligned} \lim_{r\to\infty} r\left(v_k - v_n\right) &\geq \lim_{r\to\infty} -\frac{r^2}{\kappa^2}\frac{rv_n - (\pi - c\eta)}{\left(1-\eta/\kappa\right)^4}\left(\eta/\kappa\right)^{n-k+1} \\ &\sim -\frac{r^2}{\kappa^2}\frac{c\left(\eta-\kappa\right)}{\left(1-\eta/\kappa\right)^4}\left(\eta/\kappa\right)^{n-k+1} \end{aligned}$$

Since $\left|\lim_{r\to 0} r\left(v_k - v_n\right)\right| \leq \pi$, it must be the case that $\lim_{r\to 0} r^2\left(\eta/\kappa\right)^{n-k+1}$ remain bounded; therefore $r\left(n-k\right) \sim 0$.

4. We apply Proposition A.1 to find a lower bound for $w_k - w_0$:

$$\begin{aligned} \lim_{r\to 0} r\left(w_k - w_0\right) &\geq \lim_{r\to 0}\left(\Delta w_0 + \frac{rw_0 - (\pi - 2c\eta)}{a-1}\right)\frac{ra^k}{a-1} \\ &\geq \lim_{r\to 0}\left(\frac{2c\eta - \pi}{a-1}\right)\frac{ra^k}{a-1}. \end{aligned}$$

Since $r\left(w_k - w_0\right)$ stays bounded, it must be the case that ra^k is bounded; therefore $rk \sim 0$.

Lemma A.3. $rv_{-k} \sim r\Delta v_{-k} \sim rv_{-n} \sim \Delta v_{-n} \sim 0$**.**

Proof. First, note that follower does not invest in state $k+1$ implies $c \geq \Delta v_{-(k+1)}$. We apply Proposition A.1 to find an upper bound for $(v_{-n} - v_{-k})$ as a function of rv_{-k} and $\Delta v_{-(k+1)}$:

$$v_{-n} - v_{-k} \leq \lim_{r\to 0}\left(-\Delta v_{-(k+1)}\frac{\eta}{\eta-\kappa} + (n-k)\frac{rv_{-k}}{\eta-\kappa}\right).$$

Hence, $v_{-n} - v_{-k} \leq -c\frac{\eta}{\eta-\kappa}$ and $r\left(v_{-n} - v_{-k}\right) \sim 0$.

Let $m = \text{floor}(k + \frac{n-k}{2})$. That the follower does not invest in state m implies that $c \geq \Delta v_{-m}$.

Proposition A.1. provides a lower bound for $v_{-n} - v_{-(n-1)}$ as a function of rv_{-m} and $\Delta v_{-(m+1)}$:

$$\begin{aligned}\lim_{r\to 0}\left(v_{-(n+1)} - v_{-n}\right) &\geq \lim_{r\to 0} -\Delta v_{-(m+1)}\left(\kappa/\eta\right)^{n-m} + \frac{rv_{-m} - \pi_{-m}}{\eta - \kappa} \\ &= \lim_{r\to 0}\frac{rv_{-m}}{\eta - \kappa},\end{aligned}$$

where the equality follows from $\lim_{m\to\infty}\pi_{-m} \to 0$. Hence, since the LHS is non-positive, it must be the case that $\lim_{r\to 0}\Delta v_{-n} = \lim_{r\to 0} rv_{-m} = 0$. But since $rv_{-n} \leq rv_{-m}$, it must be that $rv_{-n} \sim rv_{-k} \sim 0$. That $r\Delta v_{-k} \sim 0$ follows directly from the HJB equation for state k.

We now prove $\lim_{r\to 0} k = \infty$.

We first show that, if k is bounded, both rw_k and $r\Delta w_k$ must be asymptotically zero in order to be consistent with $rv_{-k} \sim 0$. Specifically, we use the fact that $0 \leq \pi_{-s}$ for all $0 \leq s \leq k$ and apply Proposition A.1 (simplification 1a, substituting $u_s \equiv v_{-k+1}$, $u_{s+t} = v_0$, $t = k+1$, $\Delta u_s = \Delta v_{-k}$, $a = \frac{\eta}{\eta+\kappa}$, $b = \frac{r}{\eta+\kappa}$, $\delta = \frac{rv_{-(k+1)} - (-c\eta)}{\eta+\kappa}$) to find an asymptotic upper bound for rv_0:

$$\begin{aligned}\lim_{r\to 0} rv_0 &= \lim_{r\to 0} r\left(v_0 - v_{-(k+1)}\right) \\ &\leq \lim_{r\to 0}\frac{r}{1-\kappa/\eta}\left(\Delta v_{-(k+1)} + k\frac{rv_{-(k+1)} + c\eta}{\eta}\right)\end{aligned}$$

If k is bounded, the last expression converges to zero, implying that $rv_0 \sim rw_0 \sim 0$. Lemma A.1 further implies that $\Delta w_0 \sim c$. Upper bounds for rw_k and $r\Delta w_k$ can be found, as functions of Δw_0 and rw_0, using Proposition A.1 (simplification 2, substituting $u_s \equiv w_0$, $u_{s+t} = w_k$, $t = k$, $\Delta u_s = \Delta w_0$, $a = \frac{\eta+\kappa}{\eta}$, $b = \frac{r}{\eta}$, $\delta = \frac{rw_0 - (-2c\eta)}{\eta}$):

$$\lim_{r\to 0}\left(rw_k - rw_0\right) \leq \lim_{r\to 0}\left(\Delta w_0 + \frac{rw_0 + 2c\eta}{\kappa}\right)\frac{\eta}{\kappa}r\left(\frac{\eta+\kappa}{\eta}\right)^k$$

$$\lim_{r\to 0}\left(r\Delta w_k\right) \leq \lim_{r\to 0}\left(\Delta w_0 + \frac{rw_0 + 2c\eta}{\kappa}\right)r\left(\frac{\eta+\kappa}{\eta}\right)^{k-1}. \tag{18}$$

If k is bounded, the RHS of both inequalities converge to zero, implying $rw_k \sim r\Delta w_k \sim 0$.

We now look for a contradiction. Suppose $rw_k \sim r\Delta w_k \sim 0$; we apply Proposition A.1 (simplification 1a, substituting $u_s \equiv w_k$, $u_{s+t} = w_{n+1}$, $t = n+1-k$, $\Delta u_s = \Delta w_k$, $a = \frac{\kappa}{\eta}$, $b = \frac{r}{\eta}$, $\delta = \frac{rw_k-(\pi_k-c\eta)}{\eta}$) and obtain $\frac{rw_k-(\pi_k-c\eta)}{\eta-\kappa}$ as an asymptotic upper bound for $w_{n+1} - w_n$ (noting that π_k is a lower bound for π_s for all $n \geq s \geq k$). Lemma A.2 part 2 further implies that

$$\lim_{r\to 0} \frac{rw_k - (\pi_k - c\eta)}{\eta - \kappa} \geq c$$

$$\iff \lim_{r\to 0} rw_k \geq \pi - c\kappa > 0, \tag{19}$$

which contradicts the presumption that $rw_k \sim 0$. The last inequality follows from assumption 1 ($\pi_1 - \pi_0 \geq c\kappa$), that firms is state 0 has incentive to invest when sufficiently patient. QED.

The fact that $\lim_{r\to 0} r\Delta w_k > 0$, together with inequality (18), implies $\lim_{r\to 0} r\left(\frac{\eta+\kappa}{\eta}\right)^k > 0$. We summarize these statements into a lemma, which will be useful later.

Lemma A.4. $\lim_{r\to 0} r\left(\frac{\eta+\kappa}{\eta}\right)^k > 0$ **and** $\lim_{r\to 0} r\Delta w_k > 0$**.**

A.3.3 Proof of Proposition 1.

Lemma 4 implies $g = \ln\lambda \times \left(\mu^C \cdot (\eta+\kappa) + \mu^M \cdot \kappa\right)$. We now show $\lim_{r\to 0} \alpha^{n-k}(1+\alpha)^k = 0$, which, based on Lemma 4, is a sufficient condition for $\mu^M \to 1$, $\mu^C \to 0$, and $g \to \kappa \cdot \ln\lambda$.

To proceed, we first find a lower bound for Δw_k by applying simplification 2 of Proposition A.1 (substituting $u_s \equiv w_0$, $u_{s+t} = w_k$, $t = k$, $\Delta u_s = \Delta w_0$, $a = \frac{\eta+\kappa}{\eta}$, $b = \frac{r}{\eta}$, $\delta = \frac{rw_0-(\pi-2c\eta)}{\eta}$):

$$\lim_{r\to 0} r\Delta w_k \geq C_2 \equiv \lim_{r\to 0}\left(\Delta w_0 + \frac{rw_0 - (\pi - 2c\eta)}{\kappa}\right) r\left(\frac{\eta+\kappa}{\eta}\right)^k. \tag{20}$$

Simplification 1 of Proposition A.1 provides asymptotic bounds for Δw_n (substituting $u_s \equiv w_k$, $u_{s+t} = w_n$, $t = n-k$, $\Delta u_s = \Delta w_k$, $a = \frac{\kappa}{\eta}$, $b = \frac{r}{\eta}$; the upper bound is obtained using $\delta = \frac{rw_k-(\pi_k-c\eta)}{\eta}$ and the lower bound is obtained using $\delta = \frac{rw_k-(\pi-c\eta)}{\eta}$):

$$\lim_{r\to 0}\left[\Delta w_k\left((\kappa/\eta)^{n-k} + \frac{r\eta}{(\eta-\kappa)^2}\right) + \frac{rw_k + c\eta - \pi_k}{\eta - \kappa}\right] \geq \lim_{r\to 0}\Delta w_n$$

and

$$\lim_{r\to 0}\Delta w_n \geq \lim_{r\to 0}\left[\Delta w_k\left((\kappa/\eta)^{n-k}+\frac{r\eta}{(\eta-\kappa)^2}\right)+\frac{rw_k+c\eta-\pi}{\eta-\kappa}\right].$$

Since $\lim_{r\to 0}\pi_k=\pi$, the lower and upper bounds coincide asymptotically. Furthermore, Lemma A.2 shows $\Delta w_n \sim c$; hence,

$$c \quad \sim \quad \Delta w_k\left((\kappa/\eta)^{n-k}+\frac{r\eta}{(\eta-\kappa)^2}\right)+\frac{rw_k+c\eta-\pi}{\eta-\kappa}. \tag{21}$$

Next, we apply simplification 1b of Proposition A.1 to obtain (substituting $u_s \equiv w_k$, $u_{s+t}=w_n$, $t=n-k$, $\Delta u_s=\Delta w_k$, $a=\frac{\kappa}{\eta}$, $b=\frac{r}{\eta}$; the simplification applies because $\lim_{r\to 0} r\Delta w_k > 0$, as stated in Lemma A.4):

$$r\left(w_n-w_k\right) \quad \sim \quad \frac{r\Delta w_k}{(\eta-\kappa)/\eta} \tag{22}$$

$$\implies \pi - c\kappa - rw_k \sim \frac{r\Delta w_k}{(\eta-\kappa)/\eta} \tag{23}$$

where equivalence (23) follows from part 1 of Lemma A2.

Substituting asymptotic equivalence (23) into (21), we obtain

$$c \quad \sim \quad c+\Delta w_k\left((\kappa/\eta)^{n-k}+\frac{r\eta}{(\eta-\kappa)^2}\right)-\frac{r\eta\Delta w_k}{(\eta-\kappa)^2}$$

$$\iff 0 \sim \Delta w_k\left(\kappa/\eta\right)^{n-k}$$

Inequality (20) implies

$$0 \geq \lim_{r\to 0}\left(\Delta w_0+\frac{rw_0-(\pi-2c\eta)}{\kappa}\right)\left(\frac{\eta+\kappa}{\eta}\right)^k(\kappa/\eta)^{n-k}$$

Given $\Delta w_0 \geq 0$, $rw_0 \geq 0$, and $2c\eta-\pi>0$, the inequality can hold if and only if

$$\lim_{r\to 0}\left(\frac{\eta+\kappa}{\eta}\right)^k(\kappa/\eta)^{n-k}=0,$$

as desired.

Note also that the equivalence (20) implies $r(1+\alpha)^k$ converges to a non-negative constant; hence, k grows at rate $\log r$.

A.3.4 Proof of Proposition 2.

Let (k,n) be the equilibrium investment decisions under interest rate r and (k_2,n_2) be the investments under $r-dr$. Proposition A.1 enables us to provide first-order approximations of value functions before and after the interest rate shock dr (denoted by $\{v_s\}_{s=-\infty}^{\infty}$ and $\{\hat{v}_s\}_{s=-\infty}^{\infty}$ respectively). We then use these expressions to show

$$\begin{aligned}\frac{\hat{V}^F}{V^F} &= \frac{\sum_{s=1}^{n+1}\mu_s\hat{v}_{-s}}{\sum_{s=1}^{n+1}\mu_s v_{-s}} + O(r) \\ &= \frac{k_2}{k} + O(r)\,.\end{aligned}$$

The fact that $r\left(\frac{\eta+\kappa}{\eta}\right)^k$ converges to a non-negative constant (c.f. Lemma A.4) implies

$$\frac{\hat{V}^F}{V^F} = \frac{\log(r-dr)}{\log r} + O(r)\,.$$

The part about the on-impact, proportional change in the total market value of leaders formally follows from similar derivations, but it has a more straight-forward intuition. As $r\to 0$, market leadership becomes endogenous absorbing, and the total market value of leaders becomes inversely proportional to the interest rate: $\lim_{r\to 0} rV^L = C_3 > 0$. Hence, following a decline in interest rate, the value of leaders changes proportionally with the interest rate, i.e. $\hat{V}^L/V^L = \frac{r}{r-dr} + O(r)$.

Before we prove the claim, we first establish the following lemma.

Lemma A.5. $\Delta v_{-k}\sim c$, $v_{-k}\sim\frac{c}{1-\kappa/\eta}$, $v_{-(n+1)}\sim 0$. Proof. Note that $v_{-(k-1)}-v_{-k}\geq c$, $v_{-(k-2)}-v_{-(k-1)}\geq c$, and $c\geq v_{-k}-v_{-(k+1)}$. The HJB equation for followers in state $k-1$ and k respectively imply

$$\begin{aligned} rv_{-(k-1)} &= \pi_{-(k-1)} + \eta \left(v_{-(k-2)} - v_{-(k-1)} - c \right) \\ &\quad + \kappa \left(v_{-(k-2)} - v_{-(k-1)} \right) + \eta \left(v_{-k} - v_{-(k-1)} \right) \end{aligned}$$

$$\begin{aligned} rv_{-k} &= \pi_{-k} + \eta \left(v_{-(k-1)} - v_{-k} - c \right) \\ &\quad + \kappa \left(v_{-(k-1)} - v_{-k} \right) + \eta \left(v_{-(k+1)} - v_{-k} \right). \end{aligned}$$

Substituting the previous inequalities, we get

$$rv_{-(k-1)} \leq \pi_{-(k-1)} + \kappa c + \eta \left(v_{-k} - v_{-(k-1)} \right)$$

$$rv_{-k} \geq \pi_{-k} + (\eta + \kappa) \left(v_{-(k-1)} - v_{-k} \right) - 2\eta c.$$

Hence,

$$rv_{-(k-1)} - rv_{-k} \leq \pi_{-(k-1)} - \pi_{-k} - (2\eta + \kappa) \left(v_{-(k-1)} - v_{-k} \right) + (2\eta + \kappa)\, c$$

$$\iff \left(v_{-(k-1)} - v_{-k} \right) \leq \frac{\pi_{-(k-1)} - \pi_{-k}}{2\eta + \kappa + r} + \frac{2\eta + \kappa}{2\eta + \kappa + r} c,$$

which implies $\lim_{r \to 0} \left(v_{-(k-1)} - v_{-k} \right) \leq c$. Coupled with the fact that $v_{-(k-1)} - v_{-k} \geq c$, this establishes that $v_{-(k-1)} - v_{-k} \sim c$.

We can apply simplification 1a) of Proposition A1 to show $v_{-k} - v_{-(n+1)} \sim \frac{c}{1-\kappa/\eta}$; the lemma is thus complete once we show $v_{-(n+1)} \sim 0$. Note that we can write $v_{-(n+1)}$ as a weighted average of the flow payoffs in states $k+1$ through $n+1$ and the value function in state $-k$:

$$v_{-(n+1)} = \sum_{s=k+1}^{n+1} \epsilon_s \pi_{-s} + \epsilon_k v_{-k}, \quad \text{where } \sum_{s=k}^{n} \epsilon_k = 1.$$

The flow payoffs π_{-k} approach zero as $r \to 0$; hence, $v_{-(n+1)} \sim \epsilon_k v_{-k}$. The term ϵ_k can be found by solving the recursive relationship

$$\begin{aligned}
v_{-(n+1)} &= \frac{\kappa}{r+\kappa} v_{-n} \\
v_{-n} &= \frac{\kappa}{r+\kappa+\eta} v_{-(n-1)} + \frac{\eta}{r+\kappa+\eta} v_{-(n+1)} \\
&\vdots \\
v_{-(k+1)} &= \frac{\kappa}{r+\kappa+\eta} v_{-k} + \frac{\eta}{r+\kappa+\eta} v_{-(k+2)}.
\end{aligned}$$

It is easy to see that $\epsilon_k < (\kappa/\eta)^{n-k}$; hence, as $r \to 0$, $\frac{v_{-(n+1)}}{v_{-k}} \to 0$. This implies that $v_{-(n+1)} \sim 0$ and $v_{-k} \sim \frac{c}{1-\kappa/\eta}$, as desired. QED.

We now show $\frac{\hat{V}^F}{V^F} = \frac{k_2}{k} + O(r)$. The total market value of followers is

$$\begin{aligned}
& \sum_{s=1}^{k} \mu_s v_{-s} + \sum_{s=k+1}^{n+1} \mu_s v_{-s} \\
= \; & \underbrace{2\mu_0 \left(a v_{-1} + a^2 v_{-2} + \cdots + a^k v_{-k} \right)}_{\text{total value of followers in the competitive region}} \\
& + \underbrace{\mu_{k+1} \left(v_{-(k+1)} + b v_{-(k+2)} + b^2 v_{-(k+3)} + \cdots + b^{n-k} v_{-(n+1)} \right)}_{\text{total value of followers in the monopolistic region}},
\end{aligned}$$

where $a \equiv \frac{\eta}{\eta+\kappa}$ and $b \equiv \eta/\kappa$. We analyze the two terms separately.

First, by the fact that $\Delta v_{-k} \sim c$ and $v_{-k} \sim \frac{c}{1-\alpha}$ we can apply Proposition A1 to show, for all $s \leq k$, $v_{-s} \sim \frac{c}{1-a}\left(1 - a^{k-s}\right) + \frac{ca}{1-a}\left((k-s) - \frac{a-a^{k-s}}{1-a}\right)$. Hence,

$$\begin{aligned}
\sum_{s=1}^{k} \mu_s v_{-s} &= 2\mu_0 \left(a v_{-1} + a^2 v_{-2} + \cdots + a^k v_{-k} \right) \\
&\sim 2\mu_0 \frac{c}{1-a} \sum_{s=1}^{k} \left(a^s - a^k + a\left(a^s (k-s) - \frac{a^s - a^k}{1-a} \right) \right) \\
&\sim 2\mu_0 c \left(\frac{a}{1-a} \right)^2 k,
\end{aligned}$$

where the last line follows after applying the summation formula $\sum_{s=1}^{k} (k-s) \cdot a^s = \frac{a}{1-a}\left(k - \frac{1-a^k}{1-a}\right)$.

We now compute the market value of followers in the monopolistic region. Using Proposition

A1, we derive

$$v_{-(k+s)} \sim v_{-k} - \frac{c}{1-\kappa/\eta}\left(1-(\kappa/\eta)^s\right) \sim \frac{c}{1-\kappa/\eta}(\kappa/\eta)^s,$$

thus

$$\begin{aligned} & \sum_{s=k+1}^{n+1} \mu_s v_{-s} \\ = & \ \mu_{k+1}\left(v_{-(k+1)} + (\eta/\kappa)\, v_{-(k+2)} + (\eta/\kappa)^2 v_{-(k+3)} + \cdots + (\eta/\kappa)^{n-k} v_{-(n+1)}\right) \\ \sim & \ \mu_{k+1}\frac{\alpha c}{1-\alpha}(n-k) \end{aligned}$$

The total market value of followers is thus

$$\begin{aligned} V^F & \equiv \sum_{s=1}^{k} \mu_s v_{-s} + \sum_{s=k+1}^{n+1} \mu_s v_{-s} \\ & \sim 2\mu_0 c\left(\frac{a}{1-a}\right)^2 k + \mu_{k+1}\frac{\alpha c}{1-\alpha}(n-k) \\ & = 2\mu_0\left(c\left(\frac{a}{1-a}\right)^2 k + \left(\frac{\eta}{\eta+\kappa}\right)^{k+1}\frac{\alpha c}{1-\alpha}(n-k)\right) \\ & \sim 2\mu_0 c\left(\frac{a}{1-a}\right)^2 k. \end{aligned}$$

Now consider the new equilibrium characterized (k_2, n_2) under interest rate $r - dr$. Let value functions be denoted by $\hat{v}_s$ under the new equilibrium. The market value of followers, evaluated using the steady-state under r, is

$$\hat{V}^F \equiv \sum_{s=1}^{k} \mu_s \hat{v}_{-s} + \sum_{s=k+1}^{n+1} \mu_s \hat{v}_{-s}.$$

Following the same derivation as before, we can show

$$\hat{V}^F \sim 2\mu_0 c\left(\frac{a}{1-a}\right)^2 k_2,$$

thus

$$\frac{\hat{V}^F}{V^F} = \frac{k_2}{k} + O(r),$$

as desired. That $\frac{\hat{V}^F}{V^F} = \frac{\log(r-dr)}{\log r} + O(r)$ follows from the convergence of $r\left(\frac{\eta+\kappa}{\eta}\right)^k$ to a non-negative constant (Lemma A.4.)

The on-impact, proportional change in the total market value of leaders can be derived analogously, as Proposition A.1 enables us to derive an asymptotic analytic approximation for the value functions. We omit the derivations here and instead provide a simpler intuition for the result. As interest rate converges to zero, the total market value of leaders becomes inversely proportional to the interest rate: $\lim_{r\to 0} rV^L = C_3 > 0$. Hence, following a small decline in interest rate, the value of leaders changes proportionally with the interest rate, i.e. $\hat{V}^L/V^L = \frac{r}{r-dr} + O(r)$.

B Appendix: A numerical illustration

In this numerical exercise, we relax the assumption that investments are bounded with a constant marginal cost; instead, we parametrize the investment cost as a quadratic function of the investment intensity: $c(\eta) = \delta\eta^2/2$ for $\eta \in [0,\infty)$, where δ is a cost parameter we calibrate. This is done for three reasons. First, we demonstrate numerically that Proposition 1 survive beyond the bounded and constant-marginal-cost specification. Second, a convex cost function implies that first-order conditions with respect to investments are sufficient for the model solution, reducing computational burdens. Third, the specification implies that changes in investment intensities are smoothed out across states, thereby getting around the discrete changes in investments in the "bang-bang" solution of the baseline model. All other ingredients remain unchanged from the baseline model.

The HJB equations of the numerical model follow

$$rv_s = \max_{\eta\geq 0} \pi_s - \eta^2/2 + (\kappa + \eta_{-s})(v_{s-1} - v_s) + \eta(v_{s+1} - v_s)$$

$$rv_{-s} = \max_{\eta\geq 0} \pi_{-s} - \eta^2/2 + (\kappa + \eta)\left(v_{-(s-1)} - v_{-s}\right) + \eta_s\left(v_{-(s+1)} - v_{-s}\right)$$

$$rv_0 = \max_{\eta\geq 0} \pi_0 - \eta^2/2 + \eta_0(v_{-1} - v_0) + \eta(v_1 - v_0).$$

We now provide demonstrations of the investment function $\{\eta_s\}$, the steady-state distribution

$\{\mu_s\}$, and value functions as well as how these functions change in response to lower interest rates. We also provide numerical illustrations of how steady-state levels of productivity growth vary with interest rates. In generating these numerical plots, we parametrize the within-market demand aggregator using $\sigma = \infty$, the case in which two firms produce perfect substitutes.

The top panel in Figure A2 shows the investment functions of the leader and follower across states for a high interest rate. The figure illustrates the leader dominance of Lemma 3; the leader invests more in all states beyond the neck-to-neck state. The dotted lines show the investment functions of the leader and follower for a lower interest rate. Both the leader and follower invest more in all states when the interest rate is lower, which represents the traditional effect of lower interest rates on investment.

However, as the bottom panel demonstrates, the leader's investment response to a lower interest rate is stronger than the follower's response for all states. The stronger response of the leader's investment to lower interest rates is the driving force behind the strategic effect through which lower interest rates boost market concentration.

The top panel of Figure A3 shows that, following a decline in r, the steady-state distribution of market structure shifts to the right, and aggregate market power increases.

Why does the leader's investment respond more to a lower interest rate? The bottom panel of Figure A3 shows the leader's and follower's value functions before and after a decline in the interest rate. The change in the leader's value is larger than the change in the follower's value; this is the key driver behind the leader's stronger investment response following a drop in r. Finally, Figure 1 numerically verifies the central result of the Proposition above. For a low enough interest rate, a further decline in the interest rate leads to lower growth. Figure 1 also verifies that $g \to \kappa \cdot \ln \lambda$ in the numerical exercise with variable investment intensity.

Figure 4 demonstrates Proposition 2, that declines in interest rate has asymmetric on-impact effects on the market value of leaders and followers. Starting from a high-level of interest rate, declines in r hurts leaders on average; yet, starting from a low-level of r, further declines in r unambiguously causes leaders' market value to appreciate relative to followers', and the asymmetry becomes stronger when the initial, pre-shock level of interest rate is lower.

Appendix Tables and Figures

[These tables and figures are referenced in the main text.]

Figure A1: market value of leaders respond more to decline in r, especially when initial r is low

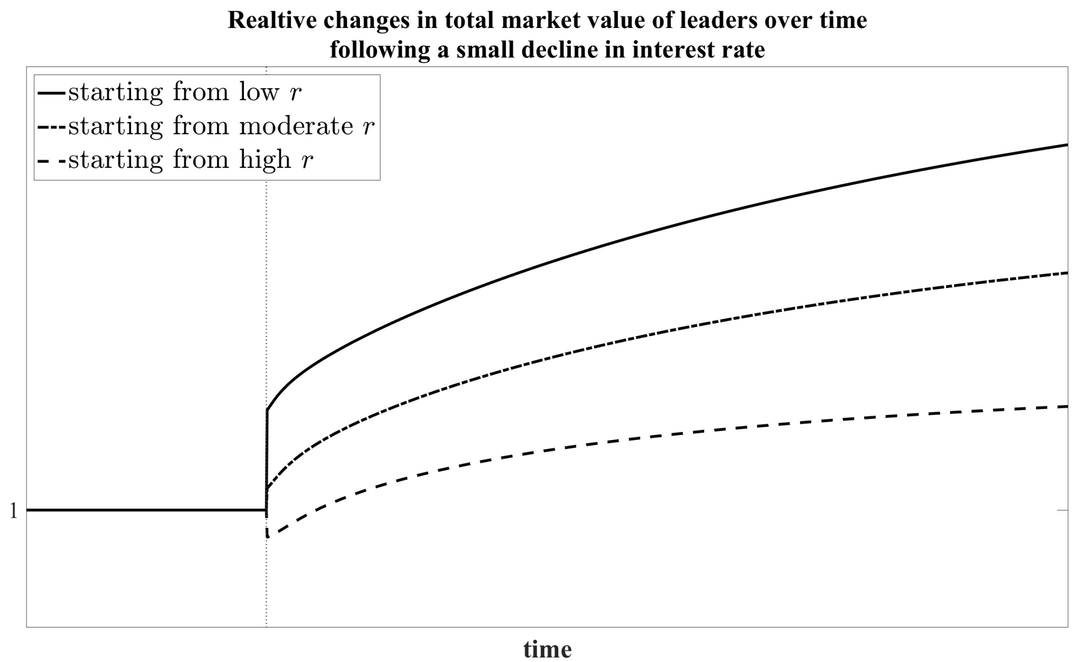


Figure A2: Investment response to a decline in r

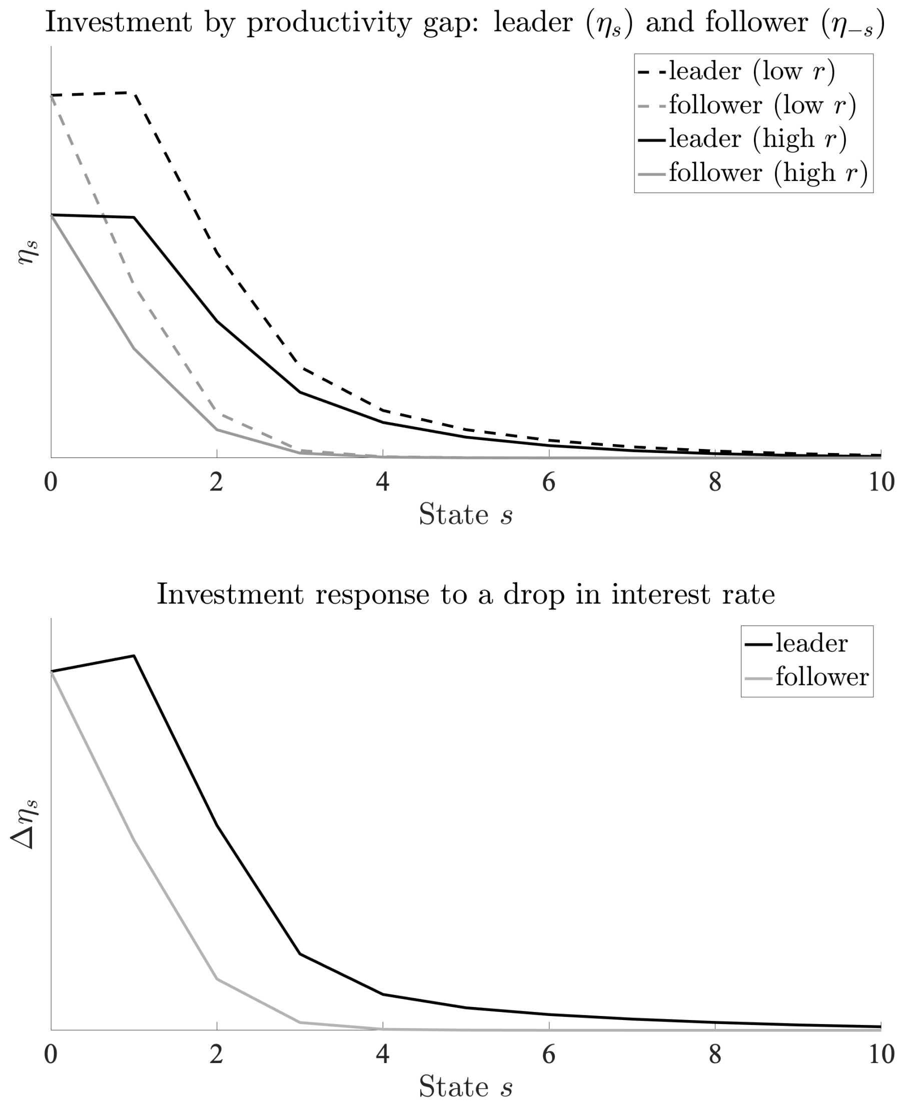


Figure A3: Response of steady-state distribution and value functions to a decline in r

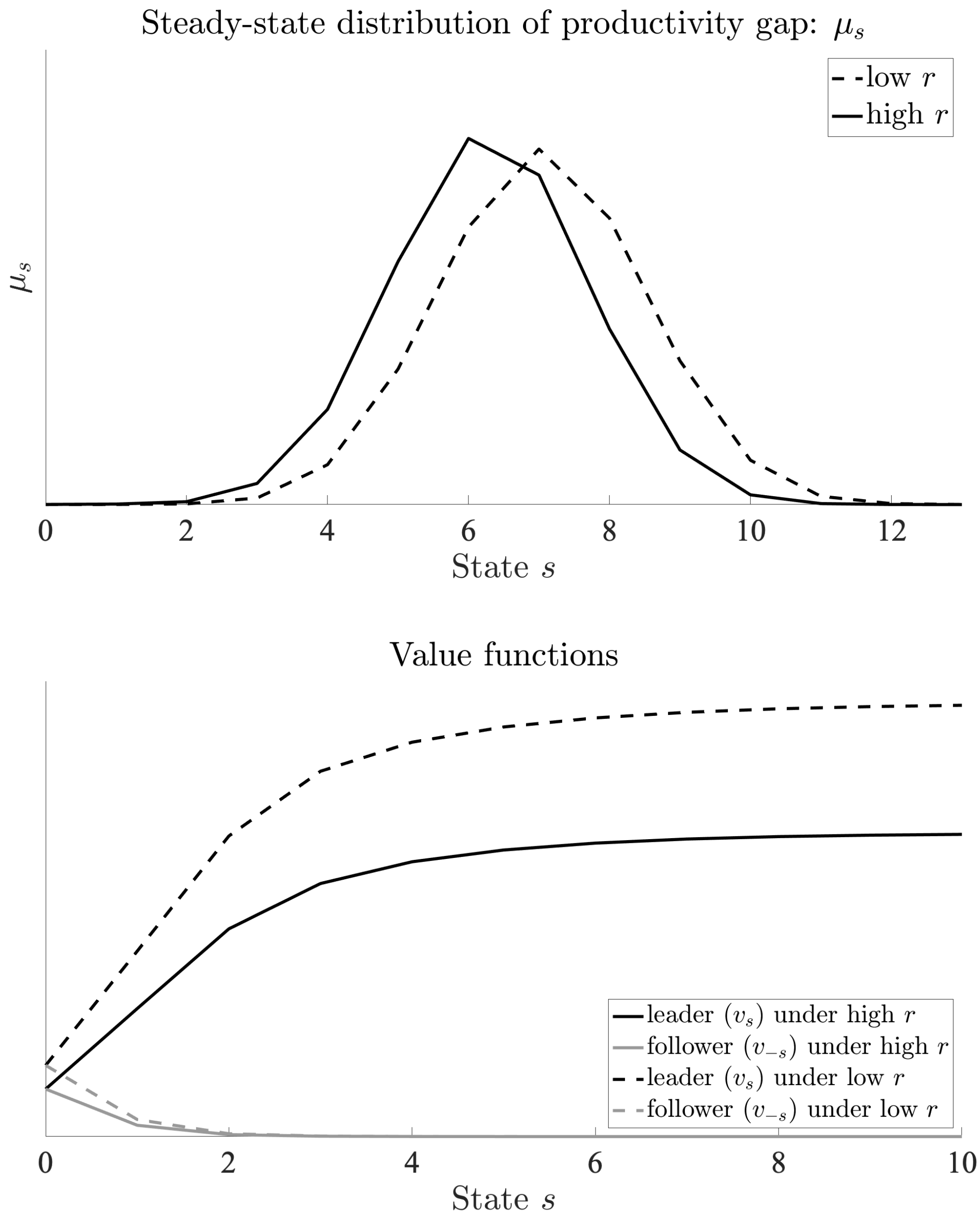


Figure A4: Aggregate profit share, market concentration and interest rate

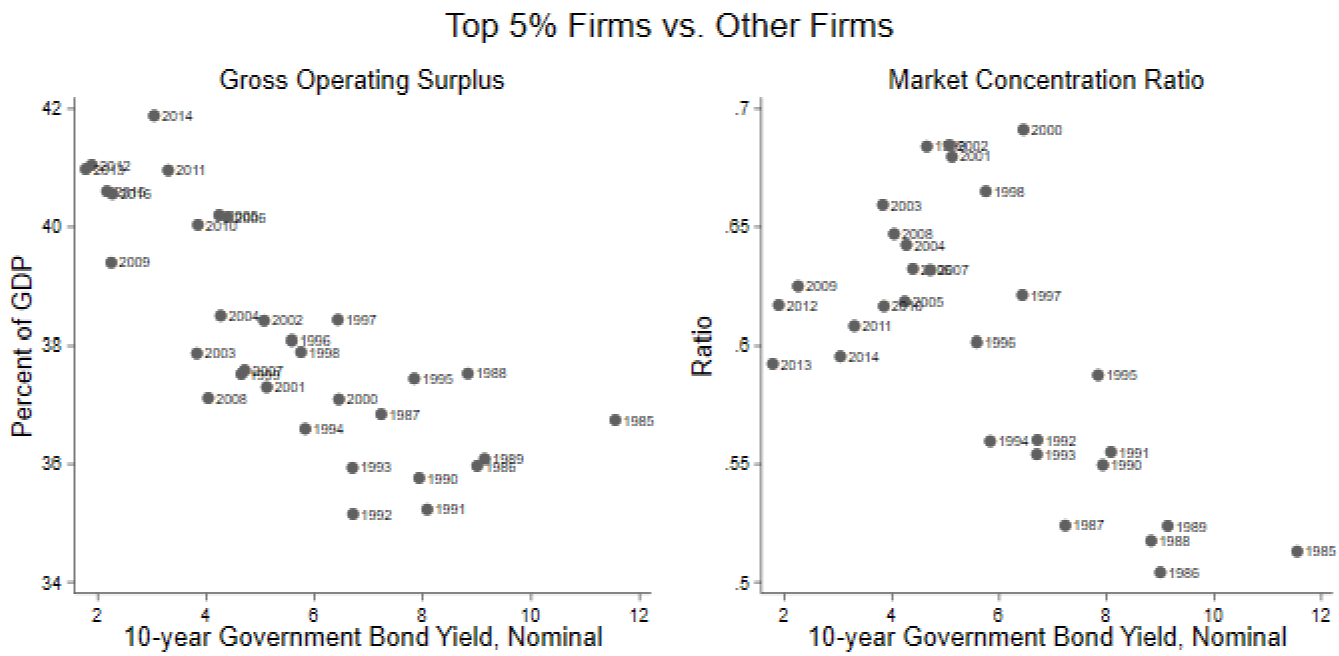


Figure A5: Business Dynamism

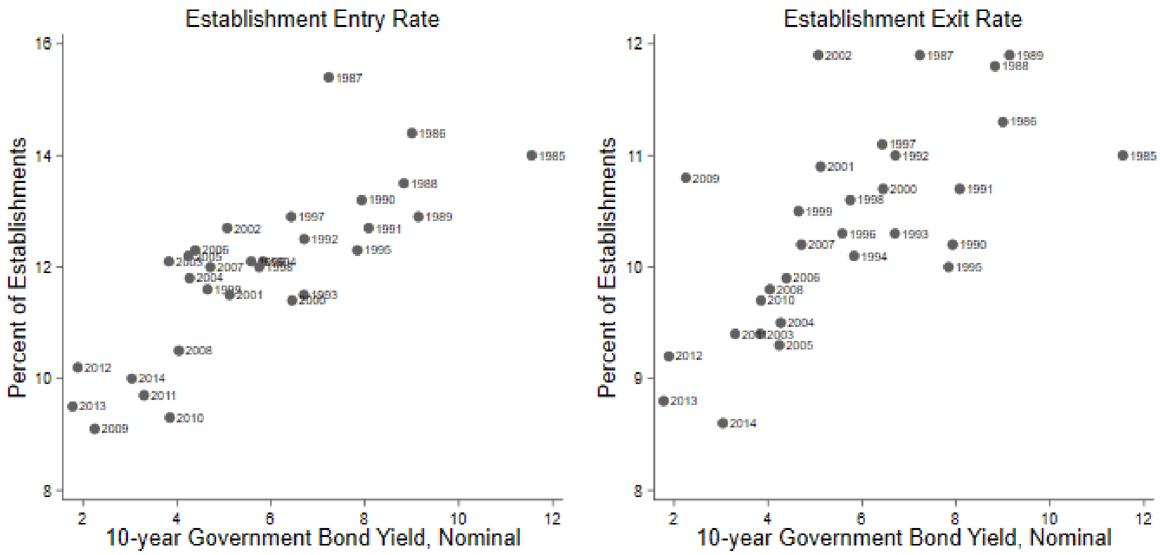
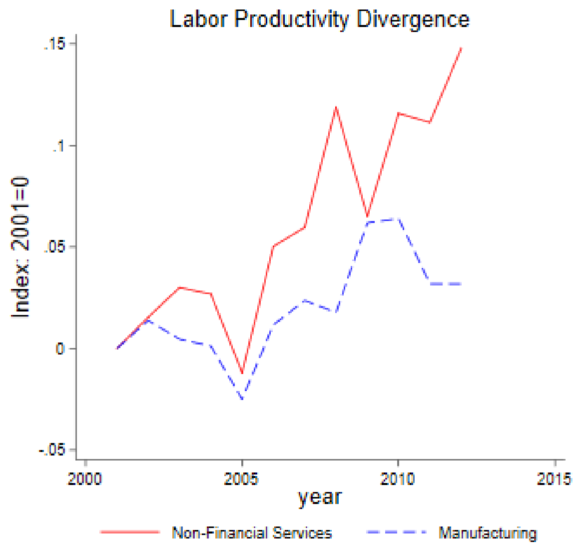


Figure A6: Widening productivity gap between leaders and followers



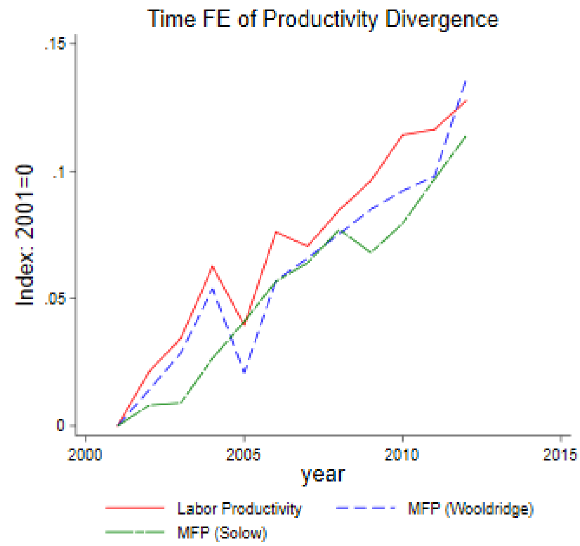


Table A1: Differential Interest Rate Responses of Leaders vs. Followers: Top 5 Percent (Full Sample)

	Stock Return					
	(1)	(2)	(3)	(4)	(5)	(6)
Top 5 Percent=1 x Δi	-1.019***	-3.303**	-4.390***	-2.183***	-3.128***	-3.493***
	(0.219)	(0.943)	(0.883)	(0.595)	(0.661)	(0.517)
Top 5 Percent=1 x Δi x Lagged i		0.254**	0.341***			0.259***
		(0.084)	(0.079)			(0.044)
Top 5 Percent=1 x Δi x Lagged real i (Clev and Fred)				0.330**	0.783***	
				(0.116)	(0.165)	
Firm β x Δi						11.74***
						(0.919)
Firm β x Δi x Lagged i						-1.096***
						(0.098)
Sample	All	All	All	All	All	All
Controls	N	N	Y	N	Y	
Industry-Date FE	Y	Y	Y	Y	Y	Y
N	74,103,576	74,103,576	46,832,612	74,103,576	46,832,612	73,745,550
R-sq	0.426	0.426	0.423	0.426	0.423	0.430

Standard errors in parentheses

* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

Regression results for the specification $\Delta \ln\left(P_{i,j,t}\right) = \alpha_{j,t} + \beta_0 D_{i,j,t} + \beta_1 D_{i,j,t}\Delta i_t + \beta_2 D_{i,j,t} i_{t-1} + \beta_3 D_{i,j,t}\Delta i_t i_{t-1} + X_{i,j,t}\gamma + \varepsilon_{i,j,t}$ for firm i in industry j at date t. $\Delta \ln\left(P_{i,j,t}\right)$ is defined here as the log change in the stock price for firm i in industry j from date $t-91$ to t (one quarter growth). $D_{i,j,t}$ is defined here as an indicator equal to 1 at date t when a firm i is in the top 5% of market capitalization in its industry j on date $t-91$. Firms with $D_{i,j,t}$=1 are called leaders while the rest are called followers. i_t is defined as the nominal 10-year Treasury yield, with i_{t-1} being the interest rate 91 days prior and Δi_t being the change in the interest rate from date $t-91$ to t. Controls X include a firm's asset-liability ratio, debt-equity ratio, book-to-market ratio, and percent of pre-tax income that goes to taxes. Industry classifications are the Fama-French industry classifications (FF). Lagged real rates were built using monthly 10-year inflation expectations from the Cleveland Fed and the daily 10-year Treasury yield at the beginning of each month (post-1982), and the CPI series from the FED (pre-1982). Standard errors are dually clustered by industry and date.

Table A2: Portfolio Returns Response to Interest Rate Changes: Top 5 Percent (Full Sample)

	Portfolio Return					
	(1)	(2)	(3)	(4)	(5)	(6)
Δi_t	-0.985***	-3.237***	-2.210***	-1.874***	-3.176***	-2.885***
	(0.277)	(0.616)	(0.497)	(0.558)	(0.909)	(0.797)
i_{t-1}		0.0597		0.00316	0.0222	0.0927
		(0.048)		(0.042)	(0.075)	(0.067)
$\Delta i_t \times i_{t-1}$		0.255***		0.0727	0.234**	0.281**
		(0.058)		(0.053)	(0.081)	(0.106)
real i_{t-1} (Clev and Fred)			0.285***			
			(0.074)			
$\Delta i_t \times$ *real* i_{t-1} (Clev and Fred)			0.344***			
			(0.103)			
Excess Market Return				-0.204***		
				(0.019)		
High Minus Low				0.0153		
				(0.037)		
$(\Delta i_t > 0)$=1 $\times \Delta i_t$					-0.103	
					(1.569)	
$(\Delta i_t > 0)$=1 $\times \Delta i_t \times i_{t-1}$					0.00546	
					(0.163)	
PE Portfolio Return						-0.272***
						(0.055)
N	13,190	13,190	13,190	13,190	13,190	10,575
R-sq	0.025	0.049	0.058	0.243	0.049	0.151

Standard errors in parentheses
* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

Regression results for the specification $R_t = \alpha + \beta_0 i_{t-1} + \beta_1 \Delta i_t + \beta_2 \Delta i_t i_{t-1} + \varepsilon_t$ at date t. R_t is defined as the market-capitalization weighted average of returns for a stock portfolio that goes long in leader stocks and goes short in follower stocks from date $t-91$ to t. Leaders are defined as the firms in the top 5% of market capitalization in its FF industry on date $t-91$. i_t is defined as the nominal 10-year Treasury yield, with i_{t-1} being the interest rate 91 days prior and Δi_t being the change in the interest rate from date $t-91$ to t. Standard errors are Newey-West with a maximum lag length of 60 days prior. Real rates were built using monthly 10-year inflation expectations from the Cleveland Fed and the daily 10-year Treasury yield (post-1982), and the CPI series from the FED (pre-1982). In column 5, the terms $(\Delta i_t > 0) = 1$ and $(\Delta i_t > 0) = 1$ x i_{t-1} were suppressed from the table. Their coefficients are 0.0222 (0.602) and -0.0616 (0.086), respectively.

Table A3: Differential Interest Rate Responses of Leaders vs. Followers: Robustness Checks

	Top 5		SIC		EBITDA		SALES	
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Top 5 Percent=1 x Δi	-1.106***	-3.847**	-1.204***	-3.903***	-1.501***	-4.805***	-1.205**	-3.684**
	(0.273)	(1.220)	(0.222)	(0.936)	(0.287)	(1.077)	(0.350)	(1.325)
Top 5 Percent=1 x Δi x Lagged i		0.303**		0.293***		0.372***		0.277*
		(0.105)		(0.081)		(0.092)		(0.112)
Sample	All	All	All	All	All	All	All	All
Industry-Date FE	Y	Y	Y	Y	Y	Y	Y	Y
N	61,313,604	61,313,604	61,277,070	61,277,070	38,957,740	38,957,740	48,247,714	48,247,714
R-sq	0.403	0.403	0.403	0.404	0.427	0.428	0.411	0.412

Standard errors in parentheses

* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

Regression results for the specification $\Delta \ln\left(P_{i,j,t}\right) = \alpha_{j,t} + \beta_0 D_{i,j,t} + \beta_1 D_{i,j,t}\Delta i_t + \beta_2 D_{i,j,t} i_{t-1} + \beta_3 D_{i,j,t}\Delta i_t i_{t-1} + X_{i,j,t}\gamma + \varepsilon_{i,j,t}$ for firm i in industry j at date t. The definitions are the same as in Table 2 except for $D_{i,j,t}$. In columns 1 and 2, leaders are chosen by the top 5 number of firms by market capitalization within an industry and date. In columns 3 and 4, leaders are chosen by the top 5% of firms by market capitalization within an industry and date, where we change the definition of industry to be the 2-digit Standard Industry Classification (SIC) codes. In columns 5 and 6, leaders are chosen by the top 5% of firms by earnings before interest, taxes, depreciation, and amortization (EBITDA) within an industry and date. In columns 7 and 8, leaders are chosen by the top 5% of firms by sales within an industry and date. Standard errors are dually clustered by industry and date.

Table A4: Portfolio Returns Response to Interest Rate Changes: Top 5 Percent, Different Frequencies

	Yearly		Semi-Yearly		Monthly		Weekly		Daily	
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
Δi_t	-1.061**	-5.570***	-1.188***	-4.594***	-1.000***	-2.365***	-0.964***	-1.846***	-0.839***	-1.244***
	(0.403)	(1.134)	(0.345)	(0.764)	(0.196)	(0.463)	(0.171)	(0.309)	(0.170)	(0.208)
i_{t-1}		0.381**		0.149		0.0273		0.00928		0.00327**
		(0.134)		(0.080)		(0.019)		(0.005)		(0.001)
$\Delta i_t \times i_{t-1}$		0.493***		0.385***		0.150***		0.0984**		0.0470
		(0.106)		(0.073)		(0.040)		(0.035)		(0.027)
Sample	All	All	All	All	All	All	All	All	All	All
N	9,037	9,037	8,962	8,962	9,081	9,081	9,099	9,099	9,080	9,080
R-sq	0.024	0.095	0.040	0.101	0.036	0.050	0.032	0.039	0.019	0.020

Standard errors in parentheses

* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

Regression results for the specification $R_t = \alpha + \beta_0 i_{t-1} + \beta_1 \Delta i_t + \beta_2 \Delta i_t i_{t-1} + \varepsilon_t$ at date t. R_t is defined as the market-capitalization weighted average of returns for a stock portfolio that goes long in leader stocks and goes short in follower stocks from date $t-91$ to t. Leaders are defined as the firms in the top 5% of market capitalization in its FF industry on date $t-J$. i_t is defined as the nominal 10-year Treasury yield, with i_{t-1} being the interest rate J days prior and Δi_t being the change in the interest rate from date $t-91$ to t. For columns 1 and 2, $J = 364$; columns 3 and 4, $J = 28$; columns 5 and 6, $J = 7$; columns 7 and 8, $J = 1$, where 1 is one trading day. Standard errors are Newey-West with a maximum lag length of 60 days prior.

Table A5: Correlation Table of Forward Rates

Variables	0-2	2-3	3-5	5-7	7-10	10-30
0-2	1.00					
2-3	0.85	1.00				
3-5	0.85	0.85	1.00			
5-7	0.80	0.76	0.67	1.00		
7-10	0.70	0.65	0.47	0.53	1.00	
10-30	0.80	0.77	0.93	0.95	0.94	1.00

Correlation table of forward rates. P-values in parentheses.