Online Appendix to *Networks, Barriers, and Trade*

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A Beyond CES

In this appendix, we show how to generalize the results in the paper beyond nested-CES functional forms.

A.1 Generalizing Sections 4 and 5 and Appendix B

In a similar vein to Baqaee and Farhi (2017a), we can extend the results in Sections 4 and 5 to arbitrary neoclassical production functions simply by replacing the input-output covariance operator with the *input-output substitution operator* instead.

For a producer k with cost function $\mathbf{C}_k$, the Allen-Uzawa elasticity of substitution between inputs x and y is

$$\theta_k(x,y) = \frac{\mathbf{C}_k d^2\mathbf{C}_k/(dp_x dp_y)}{(d\mathbf{C}_k/dp_x)(d\mathbf{C}_k/dp_y)} = \frac{\epsilon_k(x,y)}{\Omega_{ky}},$$

where $\epsilon_k(x,y)$ is the elasticity of the demand by producer k for input x with respect to the price p_y of input y, and $\tilde{\Omega}_{ky}$ is the expenditure share in cost of input y. We also use this definition for final demand aggregators.

The *input-output substitution operator* for producer k is defined as

$$\begin{aligned}\Phi_k(\Psi_{(i)},\Psi_{(j)}) &= -\sum_{x,y\in N+F} \tilde{\Omega}_{kx}[\delta_{xy} + \tilde{\Omega}_{ky}(\theta_k(x,y)-1)]\Psi_{xi}\Psi_{yj},\\ &= \frac{1}{2}E_{\Omega^{(k)}}\left((\theta_k(x,y)-1)(\Psi_i(x)-\Psi_i(y))(\Psi_j(x)-\Psi_j(y))\right),\end{aligned}$$

where δ_{xy} is the Kronecker delta, $\Psi_i(x) = \Psi_{xi}$ and $\Psi_j(x) = \Psi_{xj}$, and the expectation on the second line is over x and y.

In the CES case with elasticity θ_k, all the cross Allen-Uzawa elasticities are identical with $\theta_k(x,y) = \theta_k$ if $x \neq y$, and the own Allen-Uzawa elasticities are given by $\theta_k(x,x) = -\theta_k(1-\tilde{\Omega}_{kx})/\tilde{\Omega}_{kx}$. It is easy to verify that when $\mathbf{C}_k$ has a CES form we recover the input-output covariance operator:

$$\Phi_k(\Psi_{(i)},\Psi_{(j)}) = (\theta_k - 1)Cov_{\tilde{\Omega}^{(k)}}(\Psi_{(i)},\Psi_{(j)}).$$

Even outside the CES case, the input-output substitution operator shares many properties with the input-output covariance operator. For example, it is immediate to verify, that: $\Phi_k(\Psi_{(i)},\Psi_{(j)})$ is bilinear in $\Psi_{(i)}$ and $\Psi_{(j)}$; $\Phi_k(\Psi_{(i)},\Psi_{(j)})$ is symmetric in $\Psi_{(i)}$ and $\Psi_{(j)}$; and $\Phi_k(\Psi_{(i)},\Psi_{(j)}) = 0$ whenever $\Psi_{(i)}$ or $\Psi_{(j)}$ is a constant.

All the structural results in the paper can be extended to general non-CES economies by simply replacing terms of the form $(\theta_k - 1)Cov_{\tilde{\Omega}^{(k)}}(\Psi_{(i)}, \Psi_{(j)})$ by $\Phi_k(\Psi_{(i)}, \Psi_{(j)})$.

For example, when generalized beyond nested CES functional forms, Theorem 3 becomes the following.

Theorem 7. For a vector of perturbations to productivity $\mathrm{d}\log A$ and wedges $\mathrm{d}\log\mu$, the change in the price of a good or factor $i \in N + F$ is the same as (8). The change in the sales share of a good or factor $i \in N + F$ is

$$
\begin{aligned}
\mathrm{d}\log\lambda_i = &\sum_{k\in N+F}\left(\mathbf{1}_{\{i=k\}} - \frac{\lambda_k}{\lambda_i}\Psi_{ki}\right)\mathrm{d}\log\mu_k + \sum_{k\in N}\frac{\lambda_k}{\lambda_i}\mu_k^{-1}\Phi_k(\Psi_{(i)}, \mathrm{d}\log p) \\
&+ \sum_{g\in F^*}\sum_{c\in C}\frac{\lambda_i^{W_c} - \lambda_i}{\lambda_i}\Phi_{cg}\Lambda_g\,\mathrm{d}\log\Lambda_g,
\end{aligned}
$$

where $\mathrm{d}\log p$ is the $(N+F)\times 1$ vector of price changes in (8). The change in wedge income accruing to household c (represented by a fictitious factor) is the same as (10).

B Differential Exact-Hat Algebra

We can conduct global comparative statics by viewing Theorem 3 as a system of differential equations that can be solved by iterative means (e.g. Euler's method or Runge-Kutta). The endogenous terms in Equations (8) and (9) depend only on HAIO and Leontief matrices $(\tilde{\Omega}, \Omega, \tilde{\Psi}, \Psi)$. However, a similar logic to (9) can be used to derive changes in these matrices. In particular, the change in the HAIO matrix $\tilde{\Omega}$ is

$$
d\tilde{\Omega}_{ij} = (1-\theta_i)Cov_{\tilde{\Omega}^{(i)}}\left(d\log p, I_{(j)}\right),
$$

where $I_{(j)}$ is the jth column of the identity matrix. The change in the Leontief inverse is

$$
d\tilde{\Psi}_{ij} = \sum_{k\in N}\tilde{\Psi}_{ik}(1-\theta_i)Cov_{\tilde{\Omega}^{(k)}}\left(d\log p, \tilde{\Psi}_{(j)}\right).
$$

Similarly, changes in Ω are

$$
d\Omega_{ij} = \mu_i^{-1}d\tilde{\Omega}_{ij} - d\log\mu_i
$$

and changes in Ψ are

$$
d\Psi_{ij} = \sum_{k\in N}\Psi_{ik}\mu_k^{-1}(1-\theta_k)Cov_{\tilde{\Omega}^{(k)}}(d\log p, \Psi_{(j)}) - \sum_k \Psi_{ik}(\Psi_{kj} - \mathbf{1}_{\{k=j\}})d\log\mu_k.
$$

As explained in Appendix H, this means that we can conduct global comparative statics by repeatedly solving a $(C+F) \times (C+F)$ linear system and cumulating the results, instead of solving a system of $(C+N+F) \times (C+N+F)$ nonlinear equations. A similar approach is sometimes used in the CGE literature, for example Dixon et al. (1982), to solve high-dimensional models because exact-hat algebra is computationally impracticable for large models.[1] For the quantitative model in Section 7, the differential approach is more than ten times faster than using state-of-the-art nonlinear solvers to perform exact hat-algebra.

There are two scenarios where the differential equations approach is especially useful. The first is for large models with strong nonlinearities (e.g. low elasticities of substitution). In these cases, repeatedly solving the smaller linear system may be more computationally feasible than solving the larger highly nonlinear system.

Secondly, the differential approach is also useful outside of the nested-CES case where closed-form expressions for the demand system are not available, but estimates of the elasticity of substitution are available at different points of the cost function. In this case, the non-parametric version of Theorem 3 (Theorem 7 in Appendix A) can be used to feed estimates of the elasticity of substitution directly into the differential equation to compute global comparative statics without specifying a closed-form expression for production or cost functions.[2]

C Numerical Accuracy and Efficiency

We provide flexible Matlab code, detailed in Appendix H, that loglinearizes arbitrary general equilibrium models of the type studied in this paper and computes local comparative statics. In this section, we investigate the accuracy and computational efficiency of this approach.

Accuracy of Loglinearization. Figure 6 displays the numerical accuracy of the first-order approximation for universal iceberg and tariff shocks of different sizes. Note that this Figure 6 is not relevant for differential exact-hat algebra (as performed in Section 7) because once we iterate on the first-order approximation, it becomes exact. The left and right panels show the root-mean-squared-error in log welfare, using the benchmark model, using dollar-weighting and country-weighting. As expected, the error is larger for bigger shocks,

[1] In the CGE literature, supply and demand relationships are log-linearized and then integrated numerically by Euler's method.

[2] An additional reason why the differential equations approach can be useful is because some statistics, like real GDP, are defined in terms of path integrals. Hence, the differential equation approach *must* be used because the change in real GDP, in general, will depend on the path of integration. See Hulten (1973) for more information.

and the dollar-weighted error is smaller since nonlinearities are smaller for larger countries and less open countries.

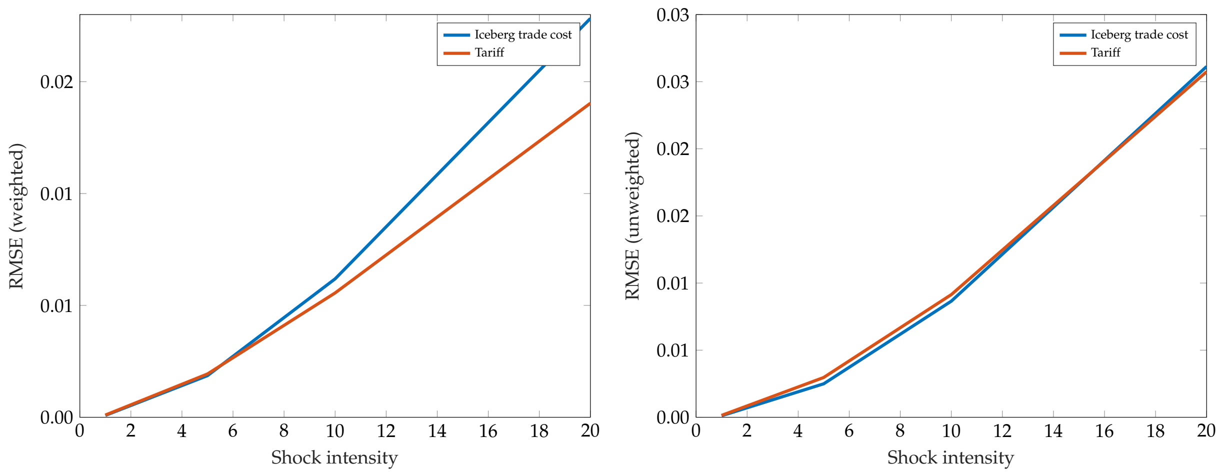


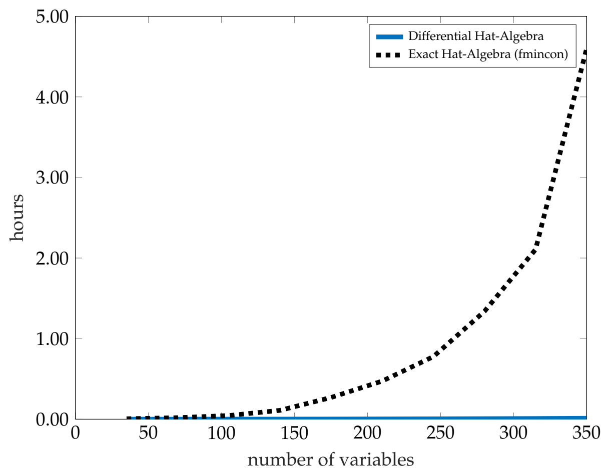
Figure 6: Error of the first-order approximation for a universal shock to trade barriers.

Computational Efficiency. By repeated iteration on the loglinear solution, the code can also compute exact nonlinear responses to shocks. We refer to this way of solving the nonlinear model as "differential exact hat-algebra." We compare the computational efficiency of differential exact hat-algebra with exact hat-algebra using Matlab's built-in *fmincon* nonlinear solver as well as a state-of-the-art industrial numerical solver Artelys *Knitro*. We provide the nonlinear solvers with analytical expressions of the Jacobian, which significantly boosts their performance. Figure 7 shows how long each solver takes to solve the model for a 60% universal increase in iceberg shocks using the benchmark elasticities. On the x-axis we vary the number of variables by varying the number of countries in descending order of country GDP. For example, when there are two countries, we only have the US and an aggregate composite "rest-of-the-world" country.[3] We increase the number of variables by disaggregating the rest-of-the-world. Figure 7 shows that differential exact hat-algebra is much faster than *fmincon* and even *Knitro*, especially as the number of countries increases.[4]

An additional virtue of the differential exact-hat algebra over standard exact-hat algebra is that when the model becomes highly nonlinear, for example when intersectoral elasticities of susbtitution are close to zero, nonlinear solvers take longer and when the domestic elasticities of substitution (θ_0, θ_1, θ_2, θ_3, θ_4) are lowered to below 0.2, *fmincon* and *Knitro* fail to find a solution at all. On the other hand, differential exact hat-algebra always

[3]Each additional country increases the number of variables by 34 — four factor and thirty goods prices.

[4]For example, the computer we used cannot solve the factor-specific version of the model using exact-hat algebra due to insufficient memory.



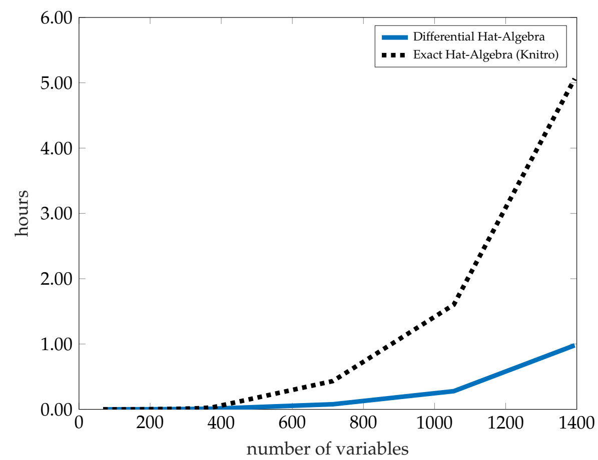


Figure 7: Time taken to solve the nonlinear model for a 60% universal iceberg shock as a function of the number of countries.

works regardless of the elasticities of substitution. This is particularly useful for large-scale applications where strong complementarities are important. For example, this algorithm is used by Bachmann et al. (2022) to study how an embargo of Russian goods would affect Germany in the short-run. This application would not have been numerically feasible using the nonlinear solvers mentioned here.

D Data Appendix

To conduct the counterfactual exercises in Section 7, we use the World Input-Output Database (Timmer et al., 2015). We use the 2013 release of the data for the final year which has no-missing data — that is 2008. We use the 2013 release because it has more detailed information on the factor usage by industry. We aggregate the 35 industries in the database to get 30 industries to eliminate missing values, and zero domestic production shares, from the data. In Table 1, we list our aggregation scheme, as well as the elasticity of substitution, based on Caliendo and Parro (2015) and taken from Costinot and Rodriguez-Clare (2014) associated with each industry. We calibrate the model to match the input-output tables and the socio-economic accounts tables in terms of expenditure shares in steady-state (before the shock).

For the growth accounting exercise in Section 7.1, we use both the 2013 and the 2016 release of the WIOD data. When we combine this data, we are able to cover a larger number of years. We compute our growth accounting decompositions for each release of the data separately, and then paste the resulting decompositions together starting with the year of overlap. To construct the consumer price index and the GDP deflator for each country, we use the final consumption weights or GDP weights of each country in each year to sum up the log price changes of each good. To arrive at the price of each good, we use the gross output prices from the socio-economic accounts tables which are reported at the (country of origin, industry) level into US dollars using the contemporaneous exchange rate, and then take log differences. This means that we assume that the log-change in the price of each good at the (origin, destination, industry of supply, industry of use) level is the same as (origin, industry of supply) level. If there are differential (changing) transportation costs over time, then this assumption is violated.

To arrive at the contemporaneous exchange rate, we use the measures of nominal GDP in the socioeconomic accounts for each year (reported in local currency) to nominal GDP in the world input-output database (reported in US dollars).

	WIOD Sector	Aggregated sector	Trade Elasticity
1	Agriculture, Hunting, Forestry and Fishing	1	8.11
2	Mining and Quarrying	2	15.72
3	Food, Beverages and Tobacco	3	2.55
4	Textiles and Textile Products	4	5.56
5	Leather, Leather and Footwear	4	5.56
6	Wood and Products of Wood and Cork	5	10.83
7	Pulp, Paper, Paper , Printing and Publishing	6	9.07
8	Coke, Refined Petroleum and Nuclear Fuel	7	51.08
9	Chemicals and Chemical Products	8	4.75
10	Rubber and Plastics	8	4.75
11	Other Non-Metallic Mineral	9	2.76
12	Basic Metals and Fabricated Metal	10	7.99
13	Machinery, Enc	11	1.52
14	Electrical and Optical Equipment	12	10.6
15	Transport Equipment	13	0.37
16	Manufacturing, Enc; Recycling	14	5
17	Electricity, Gas and Water Supply	15	5
18	Construction	16	5
19	Sale, Maintenance and Repair of Motor Vehicles...	17	5
20	Wholesale Trade and Commission Trade, ...	17	5
21	Retail Trade, Except of Motor Vehicles and...	18	5
22	Hotels and Restaurants	19	5
23	Inland Transport	20	5
24	Water Transport	21	5
25	Air Transport	22	5
26	Other Supporting and Auxiliary Transport....	23	5
27	Post and Telecommunications	24	5
28	Financial Intermediation	25	5
29	Real Estate Activities	26	5
30	Renting of M&Req and Other Business Activities	27	5
31	Public Admin/Defence; Compulsory Social Security	28	5
32	Education	29	5
33	Health and Social Work	30	5
34	Other Community, Social and Personal Services	30	5
35	Private Households with Employed Persons	30	5

Table 1: The sectors in the 2013 release of the WIOD data, and the aggregated sectors in our data.

E Factor Demand System

Adao et al. (2017) show that trading economies can be represented as if only factors are traded within and across borders, and households have preferences over factors directly. Theorem 3 can be used to flesh out this representation by locally characterizing its associated reduced-form Marshallian demand for factors in terms of sufficient-statistic microeconomic primitives. For example, in the absence of wedges, the expenditure share of household c on factor f under the "trade-in-factors" representation is given by Ψ_{cf}; the elasticities $\partial \log \Psi_{cf} / \partial \log A_i$ holding factor prices constant then characterize its Marshallian price elasticities as well as its Marshallian elasticities with respect to iceberg trade shocks:

$$\frac{\partial \log \Psi_{cf}}{\partial \log A_i} = \sum_{k \in N} \frac{\Psi_{ck}}{\Psi_{cf}} (\theta_k - 1) Cov_{\Omega^{(k)}}(\Psi_{(f)}, \Psi_{(i)}).$$

Similarly, by Theorem 3, we know that the elasticity of the factor income share of some factor j with respect to the price of another factor i, holding fixed all other factor prices, is given by

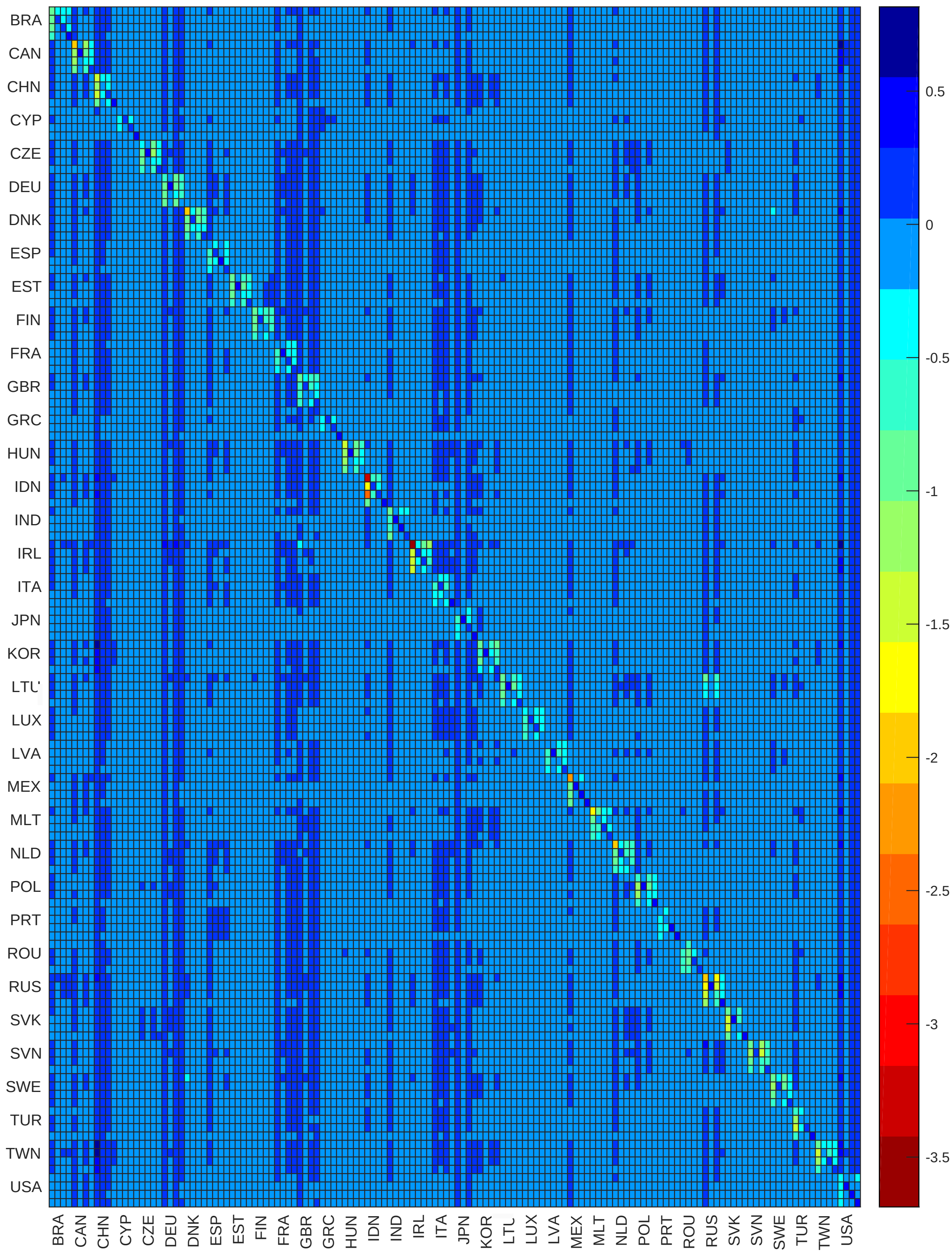
$$\frac{\partial \log \Lambda_j}{\partial \log w_i} = \sum_{k \in N} (1 - \theta_k) \frac{\lambda_k}{\Lambda_j} Cov_{\Omega^{(k)}}(\Psi_{(i)}, \Psi_{(j)}) + \sum_{c \in C} (\Lambda_j^{W_c} / \Lambda_j - 1) \Phi_{ci} \Lambda_i, \tag{21}$$

recalling that for factors $f \in F$, we interchangeably write Λ_f or λ_f to refer to their Domar weight. Figure 8 illustrates these elasticities of the factor demand system for a selection of the countries using the benchmark calibration. The ijth element gives the elasticity of j's world income share with respect to the price of i (holding fixed all other factor prices). Each country has four factors: capital, low, medium, and high skilled labor. Some interesting patterns emerge:

1. There are dark blue columns corresponding to factors in major countries like China, Germany, Britain, Japan, and the USA. For these factors, an increase in their price strongly raises the share of world income going to the rest (low-skilled labor in these countries does not obey this pattern).

2. There is a block-diagonal structure where an increase in domestic capital prices lowers both domestic labor and capital income shares. On the other hand, an increase in labor prices often raises domestic labor income and lowers domestic capital's share of world income. This is despite the fact that at the micro-level, the elasticity of substitution among domestic factors is symmetric.

Figure 8: The international factor demand system for a selection of countries



The ijth element is the elasticity of factor j with respect to the price of factor i, holding fixed other factor prices, given by equation (21).

F Aggregation and Stability of the Trade Elasticity

In this section, we characterize trade elasticities at different levels of aggregation in terms of microeconomic primitives. We also prove necessary and sufficient conditions for ensuring that the trade elasticity is constant and stable. We also relate the instability of the trade elasticity to the Cambridge Capital controversy — a mathematically similar issue that arose in capital theory in the middle of the 20th century.

F.1 Aggregating and Disaggregating Trade Elasticities

We start by defining a class of aggregate elasticities. Consider two sets of producers I and J. Let $\lambda_I = \sum_{i\in I} \lambda_i$ and $\lambda_J = \sum_{j\in J}$ be the aggregate sales shares of producers in I and J, and let $\chi_i^I = \lambda_i/\lambda_I$ and $\chi_j^J = \lambda_j/\lambda_J$. Let k be another producer. We then define the following aggregate elasticities capturing the bias towards I vs. J of a productivity shock to m as:

$$\varepsilon_{IJ,m} = \frac{\partial(\lambda_I/\lambda_J)}{\partial \log A_m},$$

where the partial derivative indicates that we allow for this elasticity to be computed holding some things constant.

To shed light on trade elasticities, we proceed as follows. Consider a set of producers $S \subseteq N_c$ in a country c. Let J be denote a set of domestic producers that sell to producers in S, and I denote a set of foreign producers that sell to producers in S. Without loss of generality, using the flexibility of network relabeling, we assume that producers in I and J are specialized in selling to producers in S so that they do not sell to producers outside of S.

Consider an iceberg trade cost modeled as a negative productivity shock $\mathrm{d}\log(1/A_m)$ to some producer m. We then define the trade elasticity as $\varepsilon_{IJ,k} = \partial(\lambda_J/\lambda_I)/\partial \log(1/A_m) = \partial(\lambda_I/\lambda_J)/\partial \log A_m$. As already mentioned, the partial derivative indicates that we allow for this elasticity to be computed holding some things constant. There are therefore different trade elasticities, depending on exactly what is held constant. Different versions of trade elasticities would be picked up by different versions of gravity equations regressions with different sorts of fixed effects and at different levels of aggregation.

There are several possibilities for what to hold constant, ranging from the most partial equilibrium to the most general equilibrium. At one extreme, we can hold constant the prices of all inputs for all the producers in I and J and the relative sales shares of all the producers in S:

$$\varepsilon_{IJ,m} = \sum_{s\in S}\sum_{i\in I}\chi_i^I(\theta_s-1)\frac{\lambda_s}{\lambda_i}Cov_{\Omega^{(s)}}(I_{(i)},\Omega_{(m)}) - \sum_{s\in S}\sum_{j\in J}\chi_j^J(\theta_s-1)\frac{\lambda_s}{\lambda_j}Cov_{\Omega^{(s)}}(I_{(j)},\Omega_{(m)}), \quad (22)$$

where $I_{(i)}$ and $I_{(j)}$ are the ith and jth columns of the identity matrix. An intermediate possibility is to hold constant the wages of all the factors in all countries:

$$\varepsilon_{IJ,k} = \sum_{i\in I}\chi_i^I\Gamma_{ik} - \sum_{j\in J}\chi_j^J\Gamma_{jk}.$$

And at the other extreme, we can compute the full general equilibrium:

$$\varepsilon_{IJ,m} = \sum_{i\in I}\chi_i^I\left(\Gamma_{im} - \sum_{g\in F}\Gamma_{ig}\frac{\mathrm{d}\log\Lambda_g}{\mathrm{d}\log A_m} + \sum_{g\in F}\Xi_{ig}\frac{\mathrm{d}\log\Lambda_g}{\mathrm{d}\log A_m}\right) - \sum_{j\in J}\chi_j^J\left(\Gamma_{jm} - \sum_{g\in F}\Gamma_{jg}\frac{\mathrm{d}\log\Lambda_g}{\mathrm{d}\log A_m} + \sum_{g\in F}\Xi_{jg}\frac{\mathrm{d}\log\Lambda_g}{\mathrm{d}\log A_m}\right),$$

$\mathrm{d}\log\Lambda_f\,/\,\mathrm{d}\log A_m$ is given in Theorem 3.

The trade elasticity is a linear combination of microeconomic elasticities of substitution, where the weights depend on the input-output structure. Except at the most microeconomic level where there is a single producer s in S and in the most partial-equilibrium setting where we recover $\epsilon_s - 1$, this means that the aggregate trade elasticity is typically an endogenous object, since the input-output structure is itself endogenous.[5] Furthermore, in the presence of input-output linkages, it is typically nonzero even for trade shocks that are not directly affecting the sales of I to J, except in the most partial-equilibrium setting.

Example: Trade Elasticity in a Round-About World Economy

In many trade models, the trade elasticity, defined holding factor wages constant, is an invariant structural parameter. As pointed out by Yi (2003), in models with intermediate inputs, the trade elasticity can easily become an endogenous object. Consider the two-country, two-good economy depicted in Figure 2. The representative household in each country only consumes the domestic good, which is produced using domestic labor and imports with a CES production function with elasticity of substitution θ. We consider the imposition of a trade cost hitting imports by country 1 from country 2. For the sake of illustration, we assume that the trade cost does not apply to the exports of country 1 to

[5] In Appendix F.3, we provide necessary and sufficient conditions for the trade elasticity to be constant in the way.

country 2.

The trade elasticity holding factor wages and foreign input prices constant is a constant structural parameter, and given simply by

$$\theta - 1.$$

However, echoing our discussion above, the trade elasticity holding factor wages constant is different, and is given by

$$\frac{\theta - 1}{1 - \Omega_{21}\Omega_{12}},$$

where Ω_{ij} is the expenditure share of i on j, e.g. its intermediate input import share. As the intermediate input shares increase, the trade elasticity becomes larger. Simple trade models without intermediate goods are incapable of generating these kinds of patterns.

Of course, since the intermediate input shares Ω_{ij} are themselves endogenous (depending on the iceberg shock), this means that the trade elasticity varies with the iceberg shocks. In particular, if $\theta > 1$, then the trade elasticity increases (nonlinearly) as iceberg costs on imports fall in all countries since intermediate input shares rise. [6]

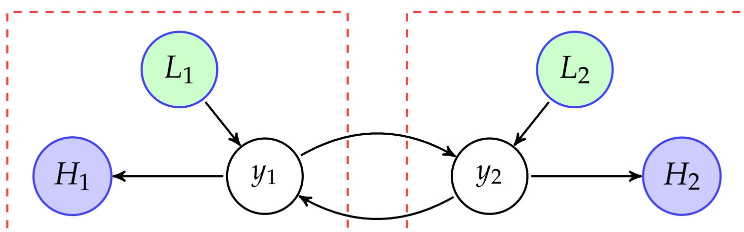


Figure 9: The solid lines show the flow of goods. Green nodes are factors, purple nodes are households, and white nodes are goods. The boundaries of each country are denoted by dashed box.

F.2 Necessary and Sufficient Conditions for Constant Trade Elasticity

In this section, we study conditions under which the trade elasticity (*holding fixed factor prices*) is constant. This trade elasticity between i and j with respect to shocks to k is defined as

$$\varepsilon_{ij,k} = \frac{\partial(\lambda_i/\lambda_j)}{\partial \log A_k},$$

[6] In Appendix F.3, we show that there it is possible to generate "trade re-switching" examples where the trade elasticity is non-monotonic with the trade cost (or even has the "wrong" sign) in otherwise perfectly respectable economies. These examples are analogous to the "capital re-switching" examples at the center the Cambridge Cambridge Capital controversy.

holding fixed factor prices. We say that a good m is *relevant* for $\varepsilon_{ij,k}$ if

$$\lambda_m Cov_{\Omega^{(m)}}(\Psi_{(k)}, \Psi_{(i)}/\lambda_i - \Psi_{(j)}/\lambda_j) \neq 0.$$

If m is not relevant, we say that it is irrelevant. For instance, if some producer m is exposed symmetrically to i and j through its inputs

$$\Omega_{ml}(\Psi_{li} - \Psi_{lj}) = 0 \quad (l \in N),$$

then $\varepsilon_{ij,k}$ is not a function of θ_m and m is irrelevant. Another example is if some producer $m \neq j$ is not exposed to k through its inputs

$$\Psi_{mk} = 0,$$

then $\varepsilon_{ij,k}$ is not a function of θ_m and m is irrelevant.

Corollary 6 (Constant Trade Elasticity)**.** *Consider two distinct goods i and j that are imported to some country c. Then consider the following conditions:*

(i) Both i and j are unconnected to one another in the production network: $\Psi_{ij} = \Psi_{ji} = 0$*, and i is not exposed to itself* $\Psi_{ii} = 1$*.*

(ii) The representative "world" household is irrelevant

$$Cov_{\chi}\left(\Psi_{(i)}, \frac{\Psi_{(i)}}{\lambda_i} - \frac{\Psi_{(j)}}{\lambda_j}\right) = 0,$$

which holds if both i and j are only used domestically, so that only household c is exposed to i and j. That is, $\lambda_i^{W_h} = \lambda_j^{W_h} = 0$ *for all* $h \neq c$*. This assumption holds automatically if i and j are imports and domestic goods and there are no input-output linkages.*

(iii) For every relevant producer l, the elasticity of substitution $\theta_l = \theta$*.*

The trade elasticity of i relative to j with respect to iceberg shocks to i is constant, and equal to

$$\varepsilon_{ij,i} = (\theta - 1).$$

if, and only if, (i)-(iii) hold.

The conditions set out in the example above, while seemingly stringent, actually represent a generalization of the conditions that hold in gravity models with constant trade elasticities. Those models oftentimes either assume away the production network, or assume that traded goods always enter via the same CES aggregator.

A noteworthy special case is when i and j are made directly from factors, without any intermediate inputs. Then, we have the following

Corollary 7. *(Network Irrelevance) If some good i and j are only made from domestic factors, then*

$$\sum_{m \in C,N} \lambda_m Cov_{\Omega^{(m)}}(\Psi_{(i)}, \Psi_{(j)}/\lambda_i - \Psi_{(i)}/\lambda_i) = 1.$$

Hence, if all microeconomic elasticities of substitution θ_m are equal to the same value $\theta_m = \theta$ then $\varepsilon_{ij,j} = \theta$.

Suppose that i is domestic goods and j are foreign imports, both of which are made only from factors (no intermediate inputs are permitted). Then a shock to j is equivalent to an iceberg shock to transportation costs. In this case, the trade elasticity of imports j into the country producing i with respect to iceberg trade costs is a convex combination of the underlying microelasticities. Of course, whenever all micro-elasticities of substitution are the same, the weights (which have to add up to one) become irrelevant, and this is the situation in most benchmark trade models with constant trade elasticities. Specifically, this highlights the fact that having common elasticities is not enough to deliver a constant trade elasticity (holding fixed factor prices) in the presence of input-output linkages as shown in the round-about example in the previous section.

F.3 Trade Reswitching

Yi (2003) shows that the trade elasticity can be nonlinear due to vertical specialization, where the trade elasticity can increase as trade barriers are lowered. Building on this insight, we can also show that, at least in principle, the trade elasticity can even have the "wrong sign" due to these nonlinearities. This relates to a parallel set of paradoxes in capital theory.

To see how this can happen, imagine there are two ways of producing a given good: the first technique uses a domestic supply chain and the other technique uses a global value chain. Whenever the good is domestically produced, the iceberg costs of transporting the good are, at most, incurred once — when the finished good is shipped to the destination. However, when the good is made via a global value chain, the iceberg costs are incurred as many times as the good is shipped across borders. As a function of the iceberg cost parameter τ, the difference in the price of these two goods (holding factor prices fixed) is a polynomial of the form

$$B_n \tau^n - B_1 \tau, \tag{23}$$

where B_n and B_1 are some coefficients and n is the number of times the border is crossed. The nonlinearity in τ, whereby the iceberg cost's effects are compounded by crossing the border, drives the sensitivity of trade volume to trade barriers in Yi (2003). The benefits from using a global value chain are compounded if the good has to cross the border many times.

However, this discussion indicates the behavior of the trade elasticity can, in principle, be much more complicated. In fact, an interesting connection can be made between the behavior of the trade elasticity and the (closed-economy) reswitching debates of the 1950s and 60s. Specifically, equation (23) is just one special case. In general, the cost difference between producing goods using supply chains of different lengths is a polynomial in τ – and this polynomial can, in principle, have more than one root. This means that the trade elasticity can be non-monotonic as a function of the trade costs, in fact, it can even have the "wrong" sign, where the volume of trade decreases as the iceberg costs fall. This mirrors the apparent paradoxes in capital theory where the relationship between the capital stock and the return on capital can be non-monotonic, and an increase in the interest rate can cause the capital stock to increase.

To see this in the trade context, imagine two perfectly substitutable goods, one of which is produced by using 10 units of foreign labor, the other is produced by shipping 1 unit of foreign labor to the home country, back to the foreign country, and then back to the home country and combining it with 10 units of domestic labor. If we normalize both foreign and domestic wages to be unity, then the costs of producing the first good is $10(1+\tau)$, whereas the cost of producing the second good is $(1+\tau)^3 + 10$, where τ is the iceberg trade cost. When $\tau = 0$, the first good dominates and goods are only shipped once across borders. When τ is sufficiently high, the cost of crossing the border is high enough that the first good again dominates. However, when τ has an intermediate value, then it can become worthwhile to produce the second good, which causes goods to be shipped across borders many times, thereby inflating the volume of trade.

Such examples are extreme, but they illustrate the point that in the presence of input-output networks, the trade elasticity even in partial equilibrium (holding factor prices constant) can behave quite unlike any microeconomic demand elasticity, sloping upwards when, at the microeconomic level, every demand curve slopes downwards.

Non-Symmetry and Non-Triviality of Trade Elasticities

Another interesting subtlety of Equation (22) is that the aggregate trade elasticities are non-symmetric. That is, in general $\varepsilon_{ij,l} \neq \varepsilon_{ji,l}$. Furthermore, unlike the standard gravity equation, Equation (22) shows that the cross-trade elasticities are, in general, nonzero. Hence, changes in trade costs between k and l can affect the volume of trade between i and j *holding fixed* relative factor prices and incomes. This is due to the presence of global value chains, which transmit shocks in one part of the economy to another independently of the usual general equilibrium effects (which work through the price of factors).

G Partial Equilibrium Counterpart to Theorem 4

Proposition 1. *For a small open economy operating in a perfectly competitive world market, the introduction of import tariffs reduces the welfare of that country's representative household by*

$$\Delta W \approx \frac{1}{2} \sum_i \lambda_i \Delta \log y_i \Delta \log \mu_i,$$

where μ_i is the ith gross tariff (no tariff is $\mu_i = 1$), y_i is the quantity of the ith import, and λ_i is the corresponding Domar weight.

Proof. To prove this, let $e(p)W$ be the expenditure function of the household. We have $e(p)W = p \cdot q + \sum_i (\mu_i - 1) p_i y_i$. Differentiate this once to get $c \cdot \mathrm{d}\, p + e(p)\, \mathrm{d}\, W = q \cdot \mathrm{d}\, p + \mathrm{d}\, q \cdot p + \sum_i \mathrm{d}\, \mu_i p_i y_i + \sum_i (\mu_i - 1)\, \mathrm{d}(p_i y_i)$. Theorem 2 implies that this can be simplified to $e(p)\, \mathrm{d}\, W = (q - c) \cdot \mathrm{d}\, p + \sum_i \mathrm{d}\, \mu_i p_i y_i + \sum_i (\mu_i - 1)\, \mathrm{d}(p_i y_i) = \sum_i (\mu_i - 1)\, \mathrm{d}(p_i y_i)$, where the left-hand side is the equivalent variation. Now differentiate this again, and evaluate at $\mu_i = 1$ to get $\sum_i p_i\, \mathrm{d}\, y_i$. Hence the second-order Taylor approximation, at $\mu = 1$, is $\frac{1}{2} \sum_i \mathrm{d}\, \mu_i p_i\, \mathrm{d}\, y_i = \frac{1}{2} \sum_i \mathrm{d} \log \mu_i p_i y_i\, \mathrm{d} \log y_i$, and our normalization implies $p_i y_i$ is equal to its Domar weight. ■

H Computational Appendix

This appendix describes our computational procedure, as well as the Matlab code in our replication files. Before running the code, customize your folder directory in the code accordingly.

Writing nested-CES economies in standard-form is useful for intuition, but it is computationally inefficient since it greatly expands the size of the input-output matrix. Therefore, for computational efficiency, we instead use the generalization in Appendix A to directly linearize the nested-CES production functions without first putting them into standard form.

Overview

First, we provide an overview of the different files before providing an in depth description of each.

1. **main_load_data.m**: First part of main code that calculates expenditure shares from data.
2. **main_dlogW.m**: Second part of the main code that loads inputs and calls functions to iterate.
3. **AES_func.m**: Function that calculates Allen-Uzawa elasticities of substitution.
4. **Nested_CES_linear_final.m**: Function that solves the system of linear equations described in Theorem 3.
5. **Nested_CES_linear_result_final.m**: Function that calculates derivatives that are used to derive welfare changes or iterate for large shocks.

While 1. and 3. are specific to our quantitative application, 2., 4. and 5. are general purpose functions that can be used to derive comparative statics and solve any model in the class we study. We now describe each part of the code in some detail.

1. Main code that loads data

Code: main_load_data.m

Data input:

1. Number of countries (C), Number of sectors in each country (N), Number of factors in each country (F)
2. Trade elasticity when a country imports or buys domestic product (`trade_elast`: N by 1 vector)
3. Input-output matrix across country and sectors (`Omega_tilde`: CN by CN matrix, (i, j) element: expenditure share of sector i on sector j)
4. Household expenditure share on heterogenous goods (`beta`: CN by C matrix, (i, c) element: expenditure share of household c on sector i)

5. Value-added share (`alpha`: CN by 1 vector, $(i, 1)$ element: value-added share of sector i), Primary Factor share (`alpha_VA`: CN by F matrix, (i, f) element: expenditure share of sector i on factor f out of factor usage)
6. A ratio of GNE of each country to world GNE (`GNE_weights`: C by 1 vector)
7. (Optional) If economy has initial tariff,
   (a) Tariff matrix when household (column) buys goods (row) – `Tariff_cons_matrix_new`: CN by C matrix ((i, c) element: tariff rate of household c, destination, on sector i, origin)
   (b) Tariff matrix when a sector (row) buys goods (column) –`Tariff_matrix_new`: CN by CN matrix ((i, j) element: tariff rate of sector i, destination, on sector j, origin)

User input:

1. If the economy does not have initial tariff, `initial_tariff_index`= 1. Otherwise, if the economy has initial tariff, =2.

Outputs:

1. `data`, shock struct

From the inputs, the code automatically calculates input shares (`beta_s`, `beta_disagg`, `Omega_s`, `Omega_disagg`, `Omega_total_C`, `Omega_total_N`) and the input-output matrix (`Omega_total_tilde`, `Omega_total`). These variables are used to calculate Allen-Uzawa elasticities of substitution and solve system of linear equations.

2. Main code that loads inputs and calls functions

Code: main_dlogW.m

Data input:

1. `data`, shock struct from **main_load_data.m**

User input:

1. Elasticity of substitution parameters for nested CES structure: Elasticity of substitution (1) across Consumption (`sigma`), (2) across Composite Value-added and Intermediates (`theta`), (3) across Primary Factors (`gamma`), and (4) across Intermediate Inputs (`epsilon`)

2. If the economy gets universal iceberg trade cost shock, `shock_index` = 1. Otherwise, if the economy gets universal tariff shock, = 2.

3. When intensity of shock is x%, `intensity` = x.

4. When shock is discretized by x/y% and model cumulates the effect of shocks y times, `ngrid` = y.

5. Ownership structure

   (a) Ownership structure of factor (`Phi_F`: C by CF matrix, (c, f) element: Factor income share of factor f owned by household c)

   (b) Ownership structure of tariff revenue (`Phi_T`: C+CN by CN+CF by C 3-D matrix, (i, j, c) element: Tariff revenue share owned by household c when household/sector i buys from sector/factor j)

6. (Optional) Technical details about how to customize iceberg trade cost shock matrix $dlog\tau$ and tariff shock matrix $dlogt$ are described in **Nested_CES_linear_final.m**

Output:

1. `dlogW` (C by `ngrid` matrix) collects change in real income of each country for each iteration of discretized shocks

2. `dlogW_sum` (C by 1 vector) shows change in real income of each country from linearized system by summing up `dlogW`

3. `dlogW_world` (1 by `ngrid` vector) is change in real income of world for each iteration of discretized shocks

4. `dlogR` (C by `ngrid` matrix) collects reallocation terms of each country for each iteration of discretized shocks

5. `dlogR_sum` (C by 1 vector) shows reallocation terms of each country from linearized system by summing up `dlogR`

6. `dlogY_2nd` shows change in world GDP to a 2nd order

3. Allen-Uzawa Elasticity of Substitution (AES)

This code computes Allen-Uzawa elasticities of substitution for each sector. These are then used following Appendix A.

Code: AES_func.m

Inputs:

1. Number of countries (C), Number of sectors in each country (N), Number of factors in each country (F)

2. Elasticity of substitution parameters for nested CES structure: Elasticity of substitution (1) across Consumption (`sigma`), (2) across Composite Value-added and Intermediates (`theta`), (3) across Primary Factors (`gamma`),and (4) across Intermediate Inputs (`epsilon`)

3. Trade elasticity when a country imports or buys domestic product (`trade_elast`: N by 1 vector)

4. Value-added share (`alpha`: CN by 1 vector, $(i, 1)$ element: value-added share of sector i)

5. Input shares:

   (a) b_{ic}(`beta_s` : C by N matrix, (c, i) element: How much household c consumes sector i good)

   (b) ω_j^{ic}(`Omega_s`: CN by N matrix, (ic, j) element: How much sector i in country c uses sector j good)

   (c) $\tilde{\Omega}_{jm}^{0c}$(`Omega_total_C` : C by CN matrix, (c, jm) element: How much household c buys from sector j in country m)

   (d) $\tilde{\Omega}_{jm}^{ic}$(`Omega_total_N` : CN by CN+CF matrix, (ic, jm) element: Hom much sector i in country c buys from good/factor j in country m)

Outputs:

1. $\theta_{0c}(ic', jm)$ (`AES_C_Mat`: CN by CN by C 3-D matrix, (ic', jm, c) element: AES of household in country c that substitutes good i in country c' and good j in country m)

2. $\theta_{kc}(ic', jm)$ (`AES_N_Mat`: CN by CN+CF by CN 3-D matrix, (ic', jm, kc) element: AES of producer of sector k in country c that substitutes good i in country c' and good/factor j in country m)

3. $\theta_{kc}(fc, jm)$ (`AES_F_Mat`: CF by CN+CF by CN 3-D matrix, (fc, jm, kc) element: AES of producer of sector k in country c that substitutes factor f in country c and good j in country m)

To describe how this code functions, we introduce the following notation.

Notation:

Let $p_{ic'}^{kc}$ be the bilateral price when industry or household k in country c buys from industry i in country c'. That is

$$p_{ic'}^{kc} = \tau_{ic'}^{kc} t_{ic'}^{kc} p_{ic'},$$

where $\tau_{ic'}^{kc}$ is an iceberg cost on kc purchasing goods from ic' and $t_{ic'}^{kc}$ is a tariff on kc purchasing goods from ic', and where $p_{ic'}$ is the marginal cost of producer i in country c'. Define

$$\Omega_{jm}^{ic} = \frac{p_{jm} x_{jm}^{ic}}{p_{ic} y_{ic}}, \quad \tilde{\Omega}_{jm}^{ic} = \frac{t_{jm}^{ic} p_{jm} x_{jm}^{ic}}{p_{ic} y_{ic}},$$

where $p_{jm} x_{jm}^{ic}$ is expenditures of ic on jm not including the import tariff. Notice that every row of $\tilde{\Omega}_{jm}^{ic}$ should always sum up to 1. Also, assume that C is a set of countries, and F_c is the factors owned by Household in country c. Then,

Households: The price of final consumption in country c

$$P_{0c} = \left(\sum_i b_{ic} \left(P_i^{0c} \right)^{1-\sigma} \right)^{\frac{1}{1-\sigma}},$$

where $b_{ic} = \sum_{m \in C} \tilde{\Omega}_{im}^{0c}$. The price of consumption good from industry i in country c

$$P_i^{0c} = \left(\sum_{m \in C} \delta_m^{0c} \left(t_{im}^{0c} \tau_{im}^{0c} p_{im} \right)^{1-\theta_i} \right)^{\frac{1}{1-\theta_i}},$$

where $\delta_m^{0c} = \tilde{\Omega}_{im}^{0c} / \left(\sum_{v \in C} \tilde{\Omega}_{iv}^{0c} \right)$.

Producers: The marginal cost of good i produced by country c

$$p_{ic} = \left(\alpha_{ic} P_{w_{ic}}^{1-\theta} + (1 - \alpha_{ic}) P_{M_{ic}}^{1-\theta} \right)^{\frac{1}{1-\theta}}$$

where $\alpha_{ic} = \sum_{f \in F_c} \tilde{\Omega}^{ic}_{fc}$. The price of value-added bundled used by producer i in country c

$$p_{w_{ic}} = \left(\sum_{f \in F_c} \alpha^{ic}_f w^{1-\gamma}_{fc} \right)^{\frac{1}{1-\gamma}},$$

where $\alpha^{ic}_f = \tilde{\Omega}^{ic}_{fc} / \left(\sum_{d \in F_c} \tilde{\Omega}^{ic}_{dc} \right)$. The price of intermediate bundle used by producer i in country c

$$p_{M_{ic}} = \left(\sum_j \omega^{ic}_j \left(q^{ic}_j \right)^{1-\varepsilon} \right)^{\frac{1}{1-\varepsilon}},$$

where $\omega^{ic}_j = \left(\sum_{m \in C} \tilde{\Omega}^{ic}_{jm} \right) / (1 - \alpha_{ic})$. The price of intermediate bundle good j used by producer i in country c

$$q^{ic}_j = \left(\sum_{m \in C} \delta^{ic}_{jm} \left(\tau^{ic}_{jm} t^{ic}_{jm} p_{jm} \right)^{1-\theta_i} \right)^{\frac{1}{1-\theta_i}},$$

where $\delta^{ic}_{jm} = \tilde{\Omega}^{ic}_{jm} / \left(\sum_{v \in C} \tilde{\Omega}^{ic}_{iv} \right)$.

Deriving Allen-Uzawa elasticities for nested-CES models is straightforward. To do so, we proceed as follows:

Derivation:

(1) $\theta_{0c}(ic', jm)$ Household demand in country c for good i from c' is

$$x^{0c}_{ic'} = \tilde{\Omega}^{0c}_{ic'} \left(\frac{p^{0c}_{ic'}}{P^{0c}_i} \right)^{-\theta_i} \left(\frac{P^{0c}_i}{P^{0c}} \right)^{-\sigma} C_c$$

Hence

$$\theta_{0c}(ic', jm) = \frac{1}{\tilde{\Omega}^{0c}_{jm}} \frac{\partial \log x^{0c}_{ic'}}{\partial \log p^{0c}_{jm}} = -\theta_i \frac{\left(\mathbf{1}(jm = ic') - \mathbf{1}(j = i) \delta^{0c}_{jm} \right)}{\tilde{\Omega}^{0c}_{jm}} - \frac{\sigma \left(\mathbf{1}(j = i) \delta^{0c}_{jm} - \tilde{\Omega}^{0c}_{jm} \right)}{\tilde{\Omega}^{0c}_{jm}}.$$

This can be simplified as

$$\theta_{0c}(ic', jm) = \frac{\theta_i}{\sum_{v \in C} \tilde{\Omega}^{0c}_{iv}} + \sigma \left(1 - \frac{1}{\sum_{v \in C} \tilde{\Omega}^{0c}_{iv}} \right) = \frac{\theta_i}{b_{ic}} + \sigma \left(1 - \frac{1}{b_{ic}} \right) \text{ when } i = j \;\&\; ic' \neq jm,$$

$$\theta_{0c}(ic', jm) = -\frac{\theta_i}{\tilde{\Omega}^{0c}_{jm}} + \frac{\theta_i}{b_{ic}} + \sigma \left(1 - \frac{1}{b_{ic}} \right) \text{ when } ic' = jm.$$

Otherwise, $\theta_{0c}(ic', jm) = \sigma$.

(2) $\theta_{kc}(ic', jm)$ When k is not a household, demand by k in country c for good i from c' is

$$x_{ic'}^{kc} = \tilde{\Omega}_{ic'}^{kc} \left(\frac{p_{ic'}^{kc}}{P_i^{kc}} \right)^{-\theta_i} \left(\frac{P_i^{kc}}{P_M^{kc}} \right)^{-\varepsilon} \left(\frac{P_M^{kc}}{p_{kc}} \right)^{-\theta} Y_{kc}.$$

Hence

$$\theta_{kc}(ic', jm) = \frac{1}{\tilde{\Omega}_{jm}^{kc}} \frac{\partial \log x_{ic'}^{kc}}{\partial \log p_{jm}^{kc}} = -\theta_i \frac{\left(\mathbf{1}(jm = ic') - \mathbf{1}(j = i)\delta_{jm}^{kc}\right)}{\tilde{\Omega}_{jm}^{kc}} - \frac{\varepsilon \left(\mathbf{1}(j = i)\delta_{jm}^{kc} - \mathbf{1}(j \notin F)\delta_{jm}^{kc}\omega_j^{kc}\right)}{\tilde{\Omega}_{jm}^{kc}} \\ - \frac{\theta \left(\mathbf{1}(j \notin F)\delta_{jm}^{kc}\omega_j^{kc} - \tilde{\Omega}_{jm}^{kc}\right)}{\tilde{\Omega}_{jm}^{kc}}.$$

This can be simplified as

$$\theta_{kc}(ic', jm) = \frac{\theta_i}{(1 - \alpha_{kc})\omega_j^{kc}} + \epsilon \left(\frac{1}{1 - \alpha_{kc}} - \frac{1}{(1 - \alpha_{kc})\omega_j^{kc}} \right) \\ + \theta \left(1 - \frac{1}{1 - \alpha_{kc}} \right) \text{ when } i = j \in N \;\&\; ic' \neq jm,$$

$$\theta_{kc}(ic', jm) = -\frac{\theta_i}{\tilde{\Omega}_{jm}^{kc}} + \frac{\theta_i}{(1 - \alpha_{kc})\omega_j^{kc}} + \epsilon \left(\frac{1}{1 - \alpha_{kc}} - \frac{1}{(1 - \alpha_{kc})\omega_j^{kc}} \right) + \theta \left(1 - \frac{1}{1 - \alpha_{kc}} \right) \text{ when } ic' = jm,$$

$$\theta_{kc}(ic', jm) = \frac{\epsilon}{1 - \alpha_{kc}} + \theta \left(1 - \frac{1}{1 - \alpha_{ic}} \right) \text{ when } i \neq j \in N,$$

and when $j \in F$, $\theta_{kc}(ic', jm) = \theta$.

(3) $\theta_{kc}(fc, jm)$ Lastly, when k is not a household, demand by k in country c for factor f is

$$x_{fc}^{kc} = \tilde{\Omega}_{fc}^{kc} \left(\frac{p_{fc}}{p_{w_{kc}}} \right)^{-\gamma} \left(\frac{p_{w_{kc}}}{p^{kc}} \right)^{-\theta} Y_{kc}.$$

Hence,

$$\theta_{kc}(fc, jm) = \frac{1}{\tilde{\Omega}_{jm}^{kc}} \frac{\partial \log x_{fc}^{kc}}{\partial \log p_{jm}^{kc}} = -\gamma \frac{\left(\mathbf{1}(jm = fc) - \mathbf{1}(jm \in F_c)\alpha_j^{ic}\right)}{\tilde{\Omega}_{jm}^{kc}} - \theta \frac{\left(\mathbf{1}(jm \in F_c)\alpha_j^{ic} - \tilde{\Omega}_{jm}^{kc}\right)}{\tilde{\Omega}_{jm}^{kc}}.$$

Notice that $\theta_{kc}(fc, jm) = \theta$ if $j \in N$. Also,

$$\theta_{kc}(fc, jc) = \frac{\gamma}{\sum_{g \in F_c} \tilde{\Omega}^{kc}_{gc}} + \theta \left(1 - \frac{1}{\sum_{g \in F_c} \tilde{\Omega}^{kc}_{gc}}\right) = \frac{\gamma}{\alpha_{kc}} + \theta \left(1 - \frac{1}{\alpha_{kc}}\right) \text{ when } j \in F \text{ \& } m = c,$$

$$\theta_{kc}(fc, jc) = -\frac{\gamma}{\tilde{\Omega}^{kc}_{fc}} + \frac{\gamma}{\alpha_{kc}} + \theta \left(1 - \frac{1}{\alpha_{kc}}\right) \text{ when } fc = jm.$$

4. Solving system of linear equations

Code: Nested_CES_linear_final.m

Input:

1. Number of countries (C), Number of sectors in each country (N), Number of factors in each country (F)
2. Allen-Uzawa elasticities of substitution:
   (a) $\theta_{0c}(ic', jm)$ (`AES_C_Mat`: CN by CN by C 3-D matrix)
   (b) $\theta_{kc}(ic', jm)$ (`AES_N_Mat`: CN by CN+CF by CN 3-D matrix)
   (c) $\theta_{kc}(fc, jm)$ (`AES_F_Mat` CF by CN+CF by CN 3-D matrix)
3. Input-output matrix and Leontief inverse
   (a) $\tilde{\Omega}^{ic}_{jm}$ (`Omega_total_tilde`: C+CN+CF by C+CN+CF matrix) : Standard form of Cost-based IO matrix
   (b) Ω^{ic}_{jm} (`Omega_total`: C+CN+CF by C+CN+CF matrix) : Standard form of Revenue-based IO matrix
   (c) $\tilde{\Psi}^{ic}_{jm}$ (`Psi_total_tilde`) : Leontief inverse of $\tilde{\Omega}^{ic}_{jm}$
   (d) Ψ^{ic}_{jm} (`Psi_total`) : Leontief inverse of Ω^{ic}_{jm}
4. Initial sales share λ_{CN} (`lambda_CN`: C+CN by 1 vector) and factor income Λ_F (`lambda_F`: CF by 1 vector)
5. Ownership structure of factor (`Phi_F`: C by CF matrix) and tariff revenue (`Phi_T`: C+CN by CN by C 3-D matrix) defined in **main_dlogW.m**
6. (Optional) If economy has initial tariff, initial tariff matrix (`init_t`: C+CN by CN matrix) defined in **main_load_data.m**

Current version of code simulates universal iceberg trade cost or tariff shock. If the user wants to specify the shocks, customize

1. universal iceberg trade cost shock matrix (`dlogtau`: C+CN by CN+CF matrix, (i, j) element: log change in iceberg trade cost when household/sector i buys from sector/factor j) or
2. tariff shock matrix (`dlogt`: C+CN by CN+CF matrix, (i, j) element: log change in tariff when household/sector i buys from sector/factor j).

Output:

Let $d\Lambda_F$ be the vector of changes in the sales of primary factors and

$$d\Lambda_{F,c',*} = \sum_{ic}\sum_{jm} \Phi_{c',ic,jm}\Omega^{ic}_{jm}(t^{ic}_{jm} - 1)d\lambda_{ic}$$

be the change in wedge-revenues of household c' due to changes in sales shares, where $\Phi_{c',ic,jm}$ is the share of tax revenues on ic's purchases of jm that go to household c'. The linear system in Theorem 3 can be written as:

$$\begin{bmatrix} d\Lambda_F \\ d\Lambda_{F_*} \end{bmatrix} = A \begin{bmatrix} d\Lambda_F \\ d\Lambda_{F_*} \end{bmatrix} + B$$

This code outputs:

1. `A` (C+CF by C+CF matrix) and `B` (C+CF by 1 vector).

Using these outputs, the code inverts the system and solves for $d\Lambda_F$(`dlambda_F`) and $d\Lambda_{F_*}$ (`dlambda_F_star`), which are used to obtain derivatives calculated by **Nested_CES_linear_result_final.m**. It updates $\tilde{\Omega}$ and other variables which are used in the next iteration.

5. Calculate derivatives

Code: Nested_CES_linear_result_final.m

Input:

All inputs used in **Nested_CES_linear_final.m** are also used in this code. Additionally, it requires

1. `GNE_weights` (C by 1 vector): A ratio of GNE of each country to world GNE
2. $d\Lambda_F$(`dlambda_F`) and $d\Lambda_{F^*}$(`dlambda_F_star`) : Solutions from **Nested_CES_linear_final.m**

Output:

1. $d\lambda$ (`dlambda_result`: C+CN+CF by 1 vector): Change in sales shares;
2. $d\chi$ (`dchi_std`: C+CN+CF by 1 vector): Change in household income shares;
3. $dlogP$ (`dlogP_Vec`: C+CN+CF by 1 vector): Change in either the price index (household), marginal cost (sector), or factor price;
4. $d\tilde{\Omega}^{ic}_{jm}$ (`dOmega_total_tilde`: C+CN+CF by C+CN+CF matrix) : Change in Cost-based IO matrix;
5. $d\Omega^{ic}_{jm}$ (`dOmega_total`: C+CN+CF by C+CN+CF matrix) : Change in Revenue-based IO matrix.

For each iteration, change in real income of country c is

$$d\log W_c = d\log \chi_c - d\log P_c$$

where $d\log P_c$ is change in price index of household c. Meanwhile, outputs are used to update $\lambda, \chi, \Omega, \tilde{\Omega}$, which are used as a simulated data with discretized shock in next iteration.

I Extension to Roy Models

Galle et al. (2017) combine a Roy-model of labor supply with an Eaton-Kortum model of trade to study the effects of trade on different groups of workers in an economy. They prove an extension to the Arkolakis et al. (2012) result that accounts for the distributional consequences of trade shocks. In this section, we show how our framework can be adapted for analyzing such models. We generalize our analysis to encompass Roy-models of the labor market, and show how duality with the closed economy can then be used to study the distributional consequences of trade.

Suppose that H_c denotes the set of households in country c. As in Galle et al. (2017), households consume the same basket of goods, but supply labor in different ways. We assume that each household type has a fixed endowment of labor L_h, which are assigned to work in different industries according to the productivity of workers in that group and the relative wage differences offered in different industries.

As usual, let world GDP be the numeraire. Define Λ_f^h to be type h's share of income derived from earning wages f

$$\Lambda_f^h = \frac{\Phi_{hf}\Lambda_f}{\chi_h},$$

where $\chi_h = \sum_{k\in F}\Phi_{hk}\Lambda_k$. The Roy model of Galle et al. (2017) implies that

$$\frac{\chi_h}{\overline{\chi_h}} = \left(\sum_f \overline{\Lambda}_f^h \left(\frac{w_f}{\overline{w}_f}\right)^{\gamma_h}\right)^{\frac{1}{\gamma_h}} \frac{L^h}{\overline{L}^h},$$

where γ_h is the supply elasticity, variables with overlines are initial values, L^h is the stock of labor h has been endowed with (since we analyze log changes, only shocks to the endowment value are relevant). Galle et al. (2017) show that the above equations can be microfounded via a model where homogenous workers in each group type draw their ability for each job from Frechet distributions, and choose to work in the job that offers them the highest return. The Roy model generalizes the factor market, with $\gamma_h = 1$ representing the case where labor cannot be moved across markets by h. If $\gamma_h > 1$ then h can take advantage of wage differentials to redirect its labor supply and boost its income. When $\gamma \to \infty$, labor mobility implies that all wages in the economy are equalized (and the model collapses to a one-factor model).

Of course, due to the fact that factor shares Λ_f^h are endogenously respond to factor prices, Theorem 3 can no longer be used to determine how these shares will change in equilibrium. Therefore, we extend those propositions here.

Proposition 2. *The response of the factor prices to a shock* $\mathrm{d}\log A_k$ *is the solution to the following system:*

1. *Product Market Equilibrium:*

$$\Lambda_l \frac{\mathrm{d}\log\Lambda_l}{\mathrm{d}\log A_k} = \sum_{j\in\{H,N\}} \lambda_j(1-\theta_j)Cov_{\Omega^{(j)}}\left(\Psi_{(k)} + \sum_f \Psi_{(f)}\frac{\mathrm{d}\log w_f}{\mathrm{d}\log A_k}, \Psi_{(l)}\right)$$
$$+ \sum_{h\in H}(\lambda_l^{W_h} - \lambda_l)\left(\sum_{f\in F_c}\Phi_{hf}\Lambda_f\frac{\mathrm{d}\log w_f}{\mathrm{d}\log A_k}\right).$$

2. *Factor Market Equilibrium:*

$$\mathrm{d}\log\Lambda_f = \sum_{h\in H} E_{\Phi^{(h)}}\left[\gamma_h\left(E_{\Lambda^{(h)}}\left(\mathrm{d}\log w_f - \mathrm{d}\log w\right)\right) + \left(E_{\Lambda^{(h)}}(\mathrm{d}\log w)\right) + (\mathrm{d}\log L)\right].$$

Given this, the welfare of the hth group is

$$\frac{\mathrm{d}\log W_h}{\mathrm{d}\log A_k} = \sum_{s\in F}\left(\Lambda_s^h - \Lambda_s^{W_h}\right)\mathrm{d}\log w_s + \lambda_k^{W_h} + \mathrm{d}\log L^h.$$

The product market equilibrium conditions are exactly the same as those in Theorem 3, but now we have some additional equations from the supply-side of the factors (which are no longer endowments). Letting $\gamma_h = 1$ for every $h \in H$ recovers Theorem 3.

J Heterogenous Households Within Countries

To extend the model to allow for a set of heterogenous agents $h \in H_c$ within country $c \in C$, we proceed as follows. We denote by H the set of all households. Each household h in country c maximizes a homogenous-of-degree-one demand aggregator

$$C_h = \mathcal{W}_h(\{c_{hi}\}_{i\in N}),$$

subject to the budget constraint

$$\sum_{i\in N} p_i c_{hi} = \sum_{f\in F}\Phi_{hf} w_f L_f + T_h,$$

where c_{hi} is the quantity of the good produced by producer i and consumed by the household, p_i is the price of good i, Φ_{hf} is the fraction of factor f owned by household, w_f is the wage of factor f, and T_h is an exogenous lump-sum transfer.

We define the following country aggregates: $c_{ci} = \sum_{h\in H_c} c_{hi}$, $\Phi_{cf} = \sum_{h\in H_c}\Phi_{hf}$, and $T_c = \sum_{h\in H_c} T_h$. We also define the HAIO matrix at the household level as a $(H+N+F)\times(H+N+F)$ matrix Ω and the Leontief inverse matrix as $\Psi = (I-\Omega)^{-1}$.

All the definitions in Section 2 remain the same. In addition, we introduce the corresponding household-level definitions for a household h. First, the nominal output and the nominal expenditure of the household are:

$$GDP_h = \sum_{f\in F}\Phi_{hf} w_f L_f, \quad GNE_h = \sum_{i\in N} p_i c_{hi} = \sum_{f\in F}\Phi_{hf} w_f L_f + T_h,$$

where we think of the household as a set producers intermediating the uses by the different producers of the different factor endowments of the household. Second, the changes in real output and real expenditure or welfare of the household are:

$$\mathrm{d}\log Y_h = \sum_{f\in F} \chi_f^{Y_h}\,\mathrm{d}\log L_f, \quad \mathrm{d}\log P_{Y_h} = \sum_{f\in F} \chi_f^{Y_h}\,\mathrm{d}\log w_f,$$

$$\mathrm{d}\log W_h = \sum_{i\in N} \chi_i^{W_h}\,\mathrm{d}\log c_{hi}, \quad \mathrm{d}\log P_{W_h} = \sum_{i\in N} \chi_i^{W_h}\,\mathrm{d}\log p_i,$$

with $\chi_f^{Y_h} = \Phi_{hf} w_f L_f / GDP_h$ and $\chi_i^{W_h} = p_i c_{hi}/GNE_h$. Third, the exposure to a good or factor k of the real expenditure and real output of household h is given by

$$\lambda_k^{W_h} = \sum_{i\in N} \chi_i^{W_h} \Psi_{ik}, \quad \lambda_k^{Y_h} = \sum_{f\in F} \chi_f^{Y_h} \Psi_{fk},$$

where recall that $\chi_i^{W_h} = p_i c_{hi}/GNE_h$ and $\chi_f^{Y_h} = \Phi_{hf} w_f L_f / GDP_h$. The exposure in real output to good or factor k has a direct connection to the sales of the producer:

$$\lambda_k^{Y_h} = 1_{\{k\in F\}} \frac{\Phi_{hk} p_k y_k}{GDP_h},$$

where $\lambda_k^{Y_h} = 1_{\{k\in F\}} \Phi_{hk} (GDP/GDP_h)\lambda_k$ the local Domar weight of k in household h and where $\Phi_{hk} = 0$ for $k \in N$ to capture the fact that the household endowment of the goods are zero. Fourth, the share of factor f in the income or expenditure of the household is given by

$$\Lambda_f^h = \frac{\Phi_{hf} w_f L_f}{GNE_h}.$$

The results in Section 3 go through without modification. Theorems 1 and 2 can be extended to the level of a household h by simply replacing the country index c by the household index h.

The results in Section 4 go through except the term on the second line of (9) must be replaced by

$$\sum_{h\in H} \frac{\lambda_i^{W_h} - \lambda_i}{\lambda_i} \Phi_{hf} \Lambda_f,$$

where we write λ_i and Λ_i interchangeably when $i \in F$ is a factor.

The results in Section 5 go through with the following changes. Theorem 4 goes through without modification, and extends to the household level where $\Delta \log Y_h \approx 0$. Theorem 5 goes through with some minor modifications. The world Bergson-Samuelson welfare function is now $W^{BS} = \sum_h \overline{\chi}_h^W \log W_h$, changes in world welfare are measured as $\Delta \log \delta$, where δ solves the equation $W^{BS}(\overline{W}_1, \ldots, \overline{W}_H) = W^{BS}(W_1/\delta, \ldots, W_H/\delta)$, where $\overline{W}_h$ are the values at the initial efficient equilibrium. We use a similar definition for country level welfare δ_c, and the same notation for household welfare δ_h. Changes in world welfare are given up to the second order by

$$\Delta \log \delta \approx \Delta \log W + Cov_{\chi_h^W} \left(\Delta \log \chi_h^W, \Delta \log P_{W_h} \right),$$

changes in country welfare are given up to the first order by

$$\Delta \log \delta_c \approx \Delta \log W_c \approx \Delta \log \chi_c^W - \Delta \log P_{W_c},$$

and the change in country welfare up to the first order by

$$\Delta \log \delta_h \approx \Delta \log W_h \approx \Delta \log \chi_h^W - \Delta \log P_{W_h}.$$

Theorems 6 goes through with some minor modifications. The final term on the last line must be replaced by

$$\frac{1}{2} \sum_{l \in N} \sum_{c \in H} \chi_c^W \Delta \log \chi_c^W \Delta \log \mu_l (\lambda_l^{W_c} - \lambda_l).$$

K Proofs

Throughout the proofs, let χ_c be the share of total world income accruing to country c.

Proof of Theorem 1. Nominal GDP is equal to

$$P_{Y_c} Y_c = \sum_{i \in N_c} (1 - 1/\mu_i) p_i y_i + \sum_{f \in F_c} w_f L_f$$

Hence

$$\begin{aligned}
d \log P_{Y_c} + d \log Y_c &= \sum_{i \in N_c} (1 - 1/\mu_i) \lambda_i^{Y_c} d \log \left((1 - 1/\mu_i) \lambda_i^{Y_c} \right) \\
&\quad + \sum_{f \in F_c} \Lambda_f^{Y_c} \left(d \log w_f + d \log L_f \right) \\
d \log Y_c &= \sum_{i \in N_c} (1 - 1/\mu_i) \lambda_i^{Y_c} d \log \left((1 - 1/\mu_i) \lambda_i^{Y_c} \right) \\
&\quad + \sum_{f \in F_c} \Lambda_f^{Y_c} \left(d \log w_f + d \log L_f \right) - d \log P^{Y_c}.
\end{aligned}$$

The price of domestic goods is given by

$$d \log p_i = d \log \mu_i - d \log A_i + \sum_{j \in N_c} \tilde{\Omega}_{ij} d \log p_j + \sum_{j \notin N_c} \tilde{\Omega}_{ij} d \log p_j,$$

which implies that

$$d \log p = (I - \tilde{\Omega}^D)^{-1} \left(d \log \mu_i - d \log A_i + \tilde{\Omega}^F \left(d \log \Lambda - d \log L \right) + \tilde{\Omega}^M d \log p^M \right),$$

where $\tilde{\Omega}^D$ is the cost-based domestic IO table, $\tilde{\Omega}^F$ are cost-based factor shares, and $\tilde{\Omega}^M$ are cost-based intermediate import shares, and $d \log p^M$ represents the change in the price of imported intermediate goods. Use the fact that

$$\begin{aligned}
d \log P_{Y_c} &= \sum_{i \in N_c} \Omega_{Y_c, i} d \log p_i - \sum_{i \in N - N_c} \Lambda_i^{Y_c} d \log p_i \\
&= \sum_{i \in N_c} \tilde{\lambda}_i^{Y_c} \left(d \log \mu_i - d \log A_i \right) + \sum_{f \in F_c} \tilde{\Lambda}_f^{Y_c} \left(d \log \Lambda_f - d \log L_f \right) \\
&\quad + \sum_{i \in N - N_c} \tilde{\Lambda}_i^{Y_c} d \log p_i - \sum_{i \in N - N_c} \Lambda_i^{Y_c} d \log p_i.
\end{aligned}$$

For an imported intermediate

$$d\log p_i = d\log \Lambda_i^{Y_c} - d\log q_i + d\log GDP$$

Substitute this back to get

$$\begin{aligned}
d\log Y_c &= \sum_{i\in N_c}(1-1/\mu_i)\lambda_i^{Y_c} d\log\left((1-1/\mu_i)\lambda_i^{Y_c}\right) + \sum_{f\in F_c}\Lambda_f^{Y_c}\left(d\log w_f + d\log L_f\right) \\
&- \sum_{i\in N_c}\tilde{\lambda}_i^{Y_c}\left(d\log\mu_i - d\log A_i\right) - \sum_{f\in F_c}\tilde{\Lambda}_f^{Y_c}\left(d\log\Lambda_f - d\log L_f\right) \\
&- \sum_{i\in N-N_c}\tilde{\Lambda}_i^{Y_c} d\log p_i + \sum_{i\in N-N_c}\Lambda_i^{Y_c} d\log p_i \\
&= \sum_{f\in F_c^*}\Lambda_f^{Y_c} d\log\Lambda_f - \sum_{i\in N_c}\tilde{\lambda}_i^{Y_c}\left(d\log\mu_i - d\log A_i\right) - \sum_{f\in F_c}\tilde{\Lambda}_f^{Y_c}\left(d\log\Lambda_f - d\log L_f\right) \\
&- \sum_{i\in N-N_c}\left(\tilde{\Lambda}_i^{Y_c} - \Lambda_i^{Y_c}\right)\left(d\log\Lambda_i^{Y_c} - d\log q_i + d\log GDP\right) \\
&= \sum_{i\in N_c}\tilde{\lambda}_i^{Y_c} d\log A_i + \sum_{f\in F_c}\tilde{\Lambda}_f^{Y_c} d\log L_f + \sum_{i\in N-N_c}\left(\tilde{\Lambda}_i^{Y_c} - \Lambda_i^{Y_c}\right) d\log q_i \\
&+ \sum_{f\in F_c^*}\Lambda_f^{Y_c}\left(d\log\Lambda_f^{Y_c} + d\log GDP_c\right) - \sum_{i\in N_c}\tilde{\lambda}_i^{Y_c} d\log\mu_i - \sum_{f\in F_c}\tilde{\Lambda}_f^{Y_c}\left(d\log\Lambda_f^{Y_c} + d\log GDP_c\right) \\
&- \sum_{i\in N-N_c}\left(\tilde{\Lambda}_i^{Y_c} - \Lambda_i^{Y_c}\right)\left(d\log\Lambda_i^{Y_c} + d\log GDP\right) \\
&= \sum_{i\in N_c}\tilde{\lambda}_i^{Y_c} d\log A_i + \sum_{f\in F_c}\tilde{\Lambda}_f^{Y_c} d\log L_f + \sum_{i\in N-N_c}\left(\tilde{\Lambda}_i^{Y_c} - \Lambda_i^{Y_c}\right) d\log q_i \\
&- \sum_{i\in N_c}\tilde{\lambda}_i^{Y_c} d\log\mu_i - \sum_{f\in F_c}\tilde{\Lambda}_f^{Y_c} d\log\Lambda_f^{Y_c} - \sum_{i\in N-N_c}\left(\tilde{\Lambda}_i^{Y_c} - \Lambda_i^{Y_c}\right)\left(d\log\Lambda_i^{Y_c}\right) \\
&+ \left[1 - \left(\sum_{f\in F_c}\tilde{\Lambda}_f^{Y_c}\right) - \sum_{i\in N-N_c}\left(\tilde{\Lambda}_i^{Y_c} - \Lambda_i^{Y_c}\right)\right] d\log GDP_c \\
&= \sum_{i\in N_c}\tilde{\lambda}_i^{Y_c} d\log A_i + \sum_{f\in F_c}\tilde{\Lambda}_f^{Y_c} d\log L_f + \sum_{i\in N-N_c}\left(\tilde{\Lambda}_i^{Y_c} - \Lambda_i^{Y_c}\right) d\log q_i \\
&- \sum_{i\in N_c}\tilde{\lambda}_i^{Y_c} d\log\mu_i - \sum_{f\in F_c}\tilde{\Lambda}_f^{Y_c} d\log\Lambda_f^{Y_c} - \sum_{i\in N-N_c}\left(\tilde{\Lambda}_i^{Y_c} - \Lambda_i^{Y_c}\right)\left(d\log\Lambda_i^{Y_c}\right).
\end{aligned}$$

The last line follows from the fact that

$$\sum_{f\in F_c}\tilde{\Lambda}_f^{Y_c} + \sum_{i\in N-N_c}\tilde{\Lambda}_i^{Y_c} = \left[1 + \sum_{i\in N-N_c}\Lambda_i^{Y_c}\right].$$

■

Proof of Theorem 2. Note that welfare is given by

$$W_c = \frac{\sum_{f \in F^*} \Phi_{cf} w_f L_f + T_c}{P^{W_c}}.$$

Hence, letting world GDP be the numeraire,

$$\mathrm{d}\log W_c = \sum_f \Lambda_f^c \left(\mathrm{d}\log \Lambda_f\right) + \frac{dT}{GNE_c} - \left(\tilde{\Omega}_{(W_c)}\right)' \mathrm{d}\log p.$$

Use the fact that

$$\mathrm{d}\log p_i = \sum_{j \in N} \tilde{\Psi}_{ij} \,\mathrm{d}\log A_j + \sum_{f \in F} \tilde{\Psi}_{if} \left(\mathrm{d}\log \Lambda_f - \mathrm{d}\log L_f\right)$$

to complete the proof. ■

Proof of Theorem 3. For each good,

$$\lambda_i = \sum_c \Omega_{W_c,i} \chi_c + \sum_i \Omega_{ji} \lambda_j,$$

where χ_c is the share of total income accruing to country c and $\Omega_{W_c,i}$ is the share of income household c spends on good i. This means

$$\lambda_i \,\mathrm{d}\log \lambda_i = \sum_c \chi_c \Omega_{W_c,i} \,\mathrm{d}\log \Omega_{W_c,i} + \sum_j \Omega_{ji} \lambda_j \,\mathrm{d}\log \Omega_{ji} + \sum_j \Omega_{ji} \,\mathrm{d}\,\lambda_j + \sum_c \Omega_{W_c,i} \chi_c \,\mathrm{d}\log \chi_c.$$

Now, note that

$$\mathrm{d}\log \Omega_{W_c,i} = (1-\theta_c)\left(\mathrm{d}\log p_i - \mathrm{d}\log P_{y_c}\right)$$

$$\mathrm{d}\log \Omega_{ji} = (1-\theta_j)\left(\mathrm{d}\log p_i - \mathrm{d}\log P_j + \mathrm{d}\log \mu_j\right) - \mathrm{d}\log \mu_j$$

$$\mathrm{d}\log \chi_c = \sum_{f \in F_c^*} \frac{\Lambda_f}{\chi_c} \,\mathrm{d}\log \Lambda_f + \sum_{i \in c} \frac{\lambda_i}{\mu_i} \,\mathrm{d}\log \mu_i.$$

$$\mathrm{d}\log p_i = \tilde{\Psi}\left(\mathrm{d}\log \mu - \mathrm{d}\log A\right) + \tilde{\Psi}\tilde{\alpha} \,\mathrm{d}\log \Lambda.$$

$$\mathrm{d}\log P_{y_c} = b'\tilde{\Psi}\left(\mathrm{d}\log \mu - \mathrm{d}\log A\right) + b'\tilde{\Psi}\tilde{\alpha} \,\mathrm{d}\log \Lambda.$$

For shock $\mathrm{d}\log \mu_k$, we have

$$\mathrm{d}\log \Omega_{W_c,i} = (1-\theta_c)\left(\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if} \,\mathrm{d}\log \Lambda_f - \sum_j \Omega_{W_c,j}\left(\tilde{\Psi}_{jk} + \sum_f \Psi_{jf} \,\mathrm{d}\log \Lambda_f\right)\right).$$

$$\mathrm{d}\log\Omega_{ji} = (1-\theta_j)\left(\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if}\,\mathrm{d}\log\Lambda_f - \tilde{\Psi}_{jk} - \sum_f \Psi_{jf}\,\mathrm{d}\log\Lambda_f\right) - \theta_j\,\mathrm{d}\log\mu_j.$$

Putting this altogether gives

$$\begin{aligned}
\mathrm{d}\,\lambda_l = & \sum_i \sum_c (1-\theta_c)\chi_c \Omega_{W_c,i}\left(\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if}\,\mathrm{d}\log\Lambda_f - \sum_j \Omega_{W_c,j}\left(\tilde{\Psi}_{jk} + \sum_f \Psi_{jf}\,\mathrm{d}\log\Lambda_f\right)\right)\Psi_{il} \\
& + \sum_i \sum_j (1-\theta_j)\lambda_j\mu_j^{-1}\tilde{\Omega}_{ji}\left(\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if}\,\mathrm{d}\log\Lambda_f - \tilde{\Psi}_{jk} - \sum_f \Psi_{jf}\,\mathrm{d}\log\Lambda_f\right)\Psi_{il} \\
& - \theta_k\lambda_k \sum_i \Omega_{ki}\Psi_{il} + \sum_c \chi_c \sum_i \Omega_{W_c,i}\Psi_{il}\,\mathrm{d}\log\chi_c.
\end{aligned}$$

Simplify this to

$$\begin{aligned}
\mathrm{d}\,\lambda_l = & \sum_c (1-\theta_c)\chi_c \Bigg[\sum_i \Omega_{W_c,i}\left(\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if}\,\mathrm{d}\log\Lambda_f\right)\Psi_{il} \\
& - \left(\sum_i \Omega_{W_c,i}\left(\tilde{\Psi}_{jk} + \sum_f \Psi_{jf}\,\mathrm{d}\log\Lambda_f\right)\right)\left(\sum_i \Omega_{W_c,i}\Psi_{il}\right)\Bigg] \\
& + \sum_j (1-\theta_j)\lambda_j\mu_j^{-1}\sum_i \tilde{\Omega}_{ji}\left(\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if}\,\mathrm{d}\log\Lambda_f\right)\Psi_{il} - \left(\sum_i \tilde{\Omega}_{ji}\Psi_{il}\right)\left(\tilde{\Psi}_{jk} + \sum_f \Psi_{jf}\,\mathrm{d}\log\Lambda_f\right) \\
& - \theta_k\lambda_k\left(\Psi_{kl} - \mathbf{1}(l=k)\right) + \sum_c \chi_c \sum_i \Omega_{W_c,i}\Psi_{il}\,\mathrm{d}\log\chi_c.
\end{aligned}$$

Simplify this further to get

$$\begin{aligned}
\mathrm{d}\,\lambda_l = & \sum_c (1-\theta_c)\chi_c Cov_{b^{(c)}}\left(\tilde{\Psi}_{(k)} + \sum_f \tilde{\Psi}_{(f)}\,\mathrm{d}\log\Lambda_f, \Psi_{(l)}\right) \\
& + \sum_j (1-\theta_j)\lambda_j\mu_j^{-1}\sum_i \tilde{\Omega}_{ji}\left(\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if}\,\mathrm{d}\log\Lambda_f\right)\Psi_{il} \\
& - \left(\sum_i \tilde{\Omega}_{ji}\Psi_{il}\right)\left(\sum_i \tilde{\Omega}_{ji}\tilde{\Psi}_{ik} + \sum_i \tilde{\Omega}_{ji}\sum_f \Psi_{if}\,\mathrm{d}\log\Lambda_f\right) \\
& - \theta_k\lambda_k\left(\Psi_{kl} - \mathbf{1}(l=k)\right) + \sum_c \chi_c \sum_i \Omega_{W_c,i}\Psi_{il}\,\mathrm{d}\log\chi_c,
\end{aligned}$$

Using the input-output covariance notation, write

$$\mathrm{d}\,\lambda_l = \sum_c (1-\theta_c)\chi_c Cov_{\Omega_{(W_c)}}\left(\tilde{\Psi}_{(k)} + \sum_f \tilde{\Psi}_{(f)}\,\mathrm{d}\log\Lambda_f, \Psi_{(l)}\right)$$

$$+\sum_{j}(1-\theta_j)\lambda_j\mu_j^{-1}Cov_{\tilde{\Omega}^{(j)}}\left(\tilde{\Psi}_{(k)}+\sum_{f}\tilde{\Psi}_{(f)}\,\mathrm{d}\log\Lambda_f,\Psi_{(l)}\right)$$

$$-(1-\theta_k)\lambda_k(\Psi_{kl}-\mathbf{1}(l=k))-\theta_k\lambda_k\left(\Psi_{kl}-\mathbf{1}(l=k)\right)+\sum_{c}\chi_c\sum_{i}\Omega_{W_c,i}\Psi_{il}\,\mathrm{d}\log\chi_c,$$

This then simplifies to give from the fact that $\sum_i \Omega_{W_c,i}\Psi_{il}=\lambda_l^{W_c}$:

$$\lambda_l\,\mathrm{d}\log\lambda_l=\sum_{j\in N,C}(1-\theta_j)\lambda_j\mu_j^{-1}Cov(\tilde{\Psi}_{(k)}+\sum_{f}^{F}\mathrm{d}\log\Lambda_f,\Psi_{(l)})$$
$$-\lambda_k\left(\Psi_{kl}-\mathbf{1}(k=l)\right)+\sum_{c}\chi_c\lambda_l^{W_c}\,\mathrm{d}\log\chi_c.$$

To complete the proof, note that

$$P_{y_c}Y_c=\sum_{f}w_fL_f+\sum_{i\in N_c}\left(1-\frac{1}{\mu_i}\right)p_iy_i.$$

Hence,

$$\mathrm{d}(P_{y_c}Y_c)=\sum_{f\in c}w_fL_f\,\mathrm{d}\log w_f+\sum_{i\in c}\left(1-\frac{1}{\mu_i}\right)p_iy_i\,\mathrm{d}\log(p_iy_i)+\sum_{i\in c}\frac{\mathrm{d}\left(1-\frac{1}{\mu_i}\right)}{\mathrm{d}\log\mu_i}p_iy_i\,\mathrm{d}\log\mu_i.$$

In other words, since $P_yY=1$, we have

$$\mathrm{d}\,\chi_c=\sum_{f\in c}\Lambda_f\,\mathrm{d}\log w_f+\sum_{i\in c}\left(1-\frac{1}{\mu_i}\right)\lambda_i\,\mathrm{d}\log\lambda_i+\sum_{i\in c}\frac{\mathrm{d}\left(1-\frac{1}{\mu_i}\right)}{\mathrm{d}\log\mu_i}\lambda_i\,\mathrm{d}\log\mu_i.$$

Hence,

$$\mathrm{d}\log\chi_c=\sum_{f\in F_c^*}\frac{\Lambda_f}{\chi_c}\,\mathrm{d}\log\Lambda_f+\sum_{i\in c}\frac{\lambda_i}{\chi_c}\,\mathrm{d}\log\mu_i.$$

■

Proof of Theorem 4. Proof of Part(1):

The expression for $\mathrm{d}^2\log Y$ follows from applying part (2) to the whole world. The equality of real GNE and real GDP at the world level completes the proof.

Proof of Part (2):

Denote the set of imports into country c by M_c. Then, we can write:

$$\frac{\mathrm{d}\log Y_c}{\mathrm{d}\log \mu_i} = \sum_{f\in F_c} \Lambda_f^{Y_c} \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_i} + \sum_j \frac{\mathrm{d}\,\lambda_j}{\mathrm{d}\log \mu_i} \frac{\left(1-\frac{1}{\mu_j}\right)}{P_{Y_c} Y_c} + \frac{\lambda_i^{Y_c}}{\mu_i} - \frac{\mathrm{d}\log P_{Y_c}}{\mathrm{d}\log \mu_i},$$

where

$$\frac{\mathrm{d}\log P_{Y_c}}{\mathrm{d}\log \mu_i} = \sum_{f\in F_c} \tilde{\Lambda}_f^{Y_c} \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_i} + \sum_{m\in M_c} \tilde{\lambda}_m^{Y_c} \frac{\mathrm{d}\log p_m}{\mathrm{d}\log \mu_i} - \tilde{\lambda}_i^{Y_c} - \sum_{m\in M_c} \Lambda_m^{Y_c} \frac{\mathrm{d}\log p_m}{\mathrm{d}\log \mu_i},$$

and

$$\tilde{\lambda}_i^{Y_c} = \sum_j \Omega_{Y_c,j} \tilde{\Psi}_{ji}.$$

Combining these expressions, we get

$$\begin{aligned}\frac{\mathrm{d}\log Y_c}{\mathrm{d}\log \mu_i} &= \sum_{f\in F_c} \left(\Lambda_f^{Y_c} - \tilde{\Lambda}_f^{Y_c}\right) \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_i} + \sum_{m\in M_c} \left(\lambda_m^{Y_c} - \tilde{\lambda}_m^{Y_c}\right) \frac{\mathrm{d}\log p_m}{\mathrm{d}\log \mu_i} \\ &\quad + \sum_{j\in N_c} \lambda_j^{Y_c} \frac{\mathrm{d}\log \lambda_j}{\mathrm{d}\log \mu_i} \left(1 - \frac{1}{\mu_j}\right) + \frac{\lambda_i^{Y_c}}{\mu_i} - \tilde{\lambda}_i^{Y_c}.\end{aligned}$$

At the efficient point,

$$\begin{aligned}\frac{\mathrm{d}^2\log Y_c}{\mathrm{d}\log \mu_i\,\mathrm{d}\log \mu_k} &= \sum_{f\in F_c} \left(\frac{\mathrm{d}\,\Lambda_f^{Y_c}}{\mathrm{d}\log \mu_i} - \frac{\mathrm{d}\,\tilde{\Lambda}_f^{Y_c}}{\mathrm{d}\log \mu_i}\right) \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k} \\ &\quad + \sum_{m\in M_c} \left(\frac{\mathrm{d}\,\lambda_m^{Y_c}}{\mathrm{d}\log \mu_i} - \frac{\mathrm{d}\,\tilde{\lambda}_m^{Y_c}}{\mathrm{d}\log \mu_i}\right) \frac{\mathrm{d}\log p_m}{\mathrm{d}\log \mu_k} - \frac{\mathrm{d}\,\tilde{\lambda}_k^{Y_c}}{\mathrm{d}\log \mu_i} \\ &\quad + \lambda_k^{Y_c} \left(\frac{\mathrm{d}\log \lambda_k^{Y_c}}{\mathrm{d}\log \mu_i} - \delta_{ki}\right) + \frac{1}{P_{Y_c} Y_c} \frac{\mathrm{d}\,\lambda_i^{Y_c}}{\mathrm{d}\log \mu_k},\end{aligned}$$

where δ_{ki} is the a Kronecker delta.

Using Lemma 9,

$$\begin{aligned}\frac{\mathrm{d}^2\log Y_c}{\mathrm{d}\log \mu_i\,\mathrm{d}\log \mu_k} &= -\sum_{f\in F_c} \lambda_i^{Y_c} \Psi_{if} \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k} - \sum_{m\in M_c} \lambda_i^{Y_c} \Psi_{im} \frac{\mathrm{d}\log p_m}{\mathrm{d}\log \mu_k} - \lambda_i^{Y_c} \left(\Psi_{ik} - \delta_{ik}\right) \\ &\quad - \lambda_k^{Y_c} \delta_{ik} + \frac{\mathrm{d}\,\lambda_i}{\mathrm{d}\log \mu_k} \frac{1}{P_{Y_c} Y_c}, \\ &= -\sum_{f\in F_c} \lambda_i^{Y_c} \Psi_{if} \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k} - \sum_{m\in M_c} \lambda_i^{Y_c} \Psi_{im} \frac{\mathrm{d}\log p_m}{\mathrm{d}\log \mu_k} - \lambda_i^{Y_c} \Psi_{ik}\end{aligned}$$

$$
\begin{aligned}
&+ \lambda_i^{Y_c} \left(\frac{\mathrm{d} \log p_i}{\mathrm{d} \log \mu_k} + \frac{\mathrm{d} \log y_i}{\mathrm{d} \log \mu_k} \right), \\
&= \lambda_i^{Y_c} \frac{\mathrm{d} \log y_i}{\mathrm{d} \log \mu_k}.
\end{aligned}
$$

■

Lemma 8. *Let χ_h be the income share of country h at the initial equilibrium. Then*

$$
\frac{\mathrm{d}\, \lambda_j}{\mathrm{d} \log \mu_k} - \sum_h \overline{\chi}_h \frac{\mathrm{d} \log \tilde{\lambda}_j^{W_h}}{\mathrm{d} \log \mu_k} = \sum_h \frac{\mathrm{d}\, \chi_h}{\mathrm{d} \log \mu_i} \lambda_j^{W_h} - \lambda_i \left(\Psi_{ij} - \delta_{ij} \right).
$$

Proof. Let μ be the diagonal matrix of μ_i and I_{μ_k} be a matrix of all zeros except μ_k for its kth diagonal element. Then

$$
\overline{\chi}' \frac{\mathrm{d}\, \tilde{\lambda}}{\mathrm{d} \log \mu_k} = \chi' \frac{d\tilde{\Omega}_{(W)}}{\mathrm{d} \log \mu_k} + \chi' \frac{\mathrm{d}\, \tilde{\lambda}}{\mathrm{d} \log \mu_k} \mu \Omega + \chi' \tilde{\lambda} I_{\mu_k} \Omega + \chi' \tilde{\lambda} \mu \frac{\mathrm{d}\, \Omega}{\mathrm{d} \log \mu_k},
$$

where $\tilde{\Omega}_{(W)}$ is a matrix whose cith element is household c's expenditure share $\tilde{\Omega}_{W_c,i}$ on good i.

On the other hand,

$$
\lambda = \chi' \tilde{\Omega}_{(W)} + \lambda \Omega.
$$

Form this, we have

$$
\frac{\mathrm{d}\, \lambda}{\mathrm{d} \log \mu_k} = \frac{\mathrm{d}\, \chi'}{\mathrm{d} \log \mu_k} \tilde{\Omega}_{(W)} + \chi' \frac{d\tilde{\Omega}_{(W)}}{\mathrm{d} \log \mu_k} + \lambda \frac{\mathrm{d}\, \Omega}{\mathrm{d} \log \mu_k} + \frac{\mathrm{d}\, \lambda}{\mathrm{d} \log \mu_k} \Omega.
$$

Combining these two expressions

$$
\left(\frac{\mathrm{d}\, \lambda}{\mathrm{d} \log \mu_k} - \overline{\chi}' \frac{\mathrm{d} \log \tilde{\lambda}}{\mathrm{d} \log \mu_k} \right) = \left(\frac{\mathrm{d}\, \lambda}{\mathrm{d} \log \mu_k} - \overline{\chi}' \frac{\mathrm{d} \log \tilde{\lambda}}{\mathrm{d} \log \mu_k} \right) \Omega + \frac{\mathrm{d}\, \chi}{\mathrm{d} \log \mu_k} \tilde{\Omega}_{(W)} - \chi' \tilde{\lambda}^{(h)} I_{\mu_k} \Omega.
$$

Rearrange this to get

$$
\left(\frac{\mathrm{d}\, \lambda}{\mathrm{d} \log \mu_k} - \overline{\chi}' \frac{\mathrm{d} \log \tilde{\lambda}}{\mathrm{d} \log \mu_k} \right) = \frac{\mathrm{d}\, \chi}{\mathrm{d} \log \mu_k} \tilde{\Omega}_{(W)} \Psi - \chi' \tilde{\lambda}^{(h)} I_{\mu_k} (\Psi - I),
$$

or

$$
\left(\frac{\mathrm{d}\, \lambda}{\mathrm{d} \log \mu_k} - \overline{\chi}' \frac{\mathrm{d} \log \tilde{\lambda}}{\mathrm{d} \log \mu_k} \right) = \frac{\mathrm{d}\, \chi}{\mathrm{d} \log \mu_k} \tilde{\Omega}_{(W)} \Psi - \lambda I_{\mu_k} (\Psi - I).
$$

■

Lemma 9. *At the efficient steady-state*

$$\frac{d\,\lambda_j^{Y_c}}{d\log\mu_k}-\frac{d\,\tilde{\lambda}_j^{Y_c}}{d\log\mu_k}=-\lambda_k^{Y_c}\left(\Psi_{kj}-\delta_{kj}\right).$$

Proof. Start from the relations

$$\lambda_j^{Y_c}=\chi_j^{Y_c}+\sum_i\lambda_i^{Y_c}\Omega_{ij},$$

and

$$\tilde{\lambda}_j^{Y_c}=\chi_j^{Y_c}+\sum_i\tilde{\lambda}_i^{Y_c}\mu_i\Omega_{ij}.$$

Differentiate both to get

$$\frac{d\,\lambda_j^{Y_c}}{d\log\mu_k}-\frac{d\,\tilde{\lambda}_j^{Y_c}}{d\log\mu_k}=\sum_i\left(\frac{d\,\lambda_j^{Y_c}}{d\log\mu_k}-\frac{d\,\tilde{\lambda}_j^{Y_c}}{d\log\mu_k}\right)\Omega_{ij}-\lambda_k^{Y_c}\Omega_{ki}.$$

Rearrange this to get the desired result. ∎

Proof of Corollary 5. Let $\overline{\chi}_h^W$ be the elasticity of social welfare with respect to the consumption of country h (i.e. log Pareto weight). Then

$$\frac{d\log W^{BS}}{d\log\mu_k}=\sum_{h\in H}\overline{\chi}_h^W\frac{d\log W_h}{d\log\mu_k}=\sum_h\overline{\chi}_h^W\left(\frac{d\log\chi_h^W}{d\log\mu_k}-\frac{d\log P_{cpi,h}}{d\log\mu_k}\right).$$

$$\frac{d\log\chi_h^W}{d\log\mu_k}=\sum_{f\in F_c}\frac{\Lambda_f}{\chi_h}\frac{d\log\Lambda_f}{d\log\mu_k}+\sum_{i\in N_h}\frac{d\,\lambda_i}{d\log\mu_k}\frac{(1-\frac{1}{\mu_i})}{\chi_h}.$$

$$\frac{d\log P_{cpi,h}}{d\log\mu_k}=\sum_{f\in F}\tilde{\Lambda}_f^{W_h}\frac{d\log\Lambda_f}{d\log\mu_k}+\tilde{\lambda}_k^{W_h}.$$

Hence, assuming the normalization $P_YY=1$ gives

$$\begin{aligned}\frac{d^2\log W^{BS}}{d\log\mu_k\,d\log\mu_i}=\sum_h\overline{\chi}_h^W&\Bigg(\sum_f\frac{d\,\Lambda_f}{d\log\mu_i}\frac{d\log\Lambda_f}{d\log\mu_k}\frac{1}{\chi_h^W}+\sum_f\frac{\Lambda_f}{\chi_h^W}\frac{d^2\log\Lambda_f}{d\log\mu_i\,d\log\mu_k}\\&-\sum_f\frac{\Lambda_f}{\chi_h^W}\frac{d\log\Lambda_f}{d\log\mu_k}\frac{d\log\chi_h^W}{d\log\mu_i}+\frac{d\,\lambda_k}{d\log\mu_i}\frac{1}{\chi_h^W\mu_k}-\frac{\lambda_k}{\chi_h^W\mu_k}\frac{d\log\chi_h^W}{d\log\mu_i}-\frac{\lambda_k}{\chi_h^W\mu_k}\delta_{ki}\\&\sum_i\frac{d^2\,\lambda_j}{d\log\mu_i\,d\log\mu_k}\frac{1-\frac{1}{\mu_j}}{\chi_h}+\frac{d\,\lambda_i}{d\log\mu_k}\frac{1}{\mu_i\chi_h^W}+\sum_j\frac{d\,\lambda_j}{d\log\mu_k}\frac{1-\frac{1}{\mu_j}}{\chi_h^W}\frac{d\log\chi_h^W}{d\log\mu_i}\end{aligned}$$

$$-\sum_f \frac{\mathrm{d}\,\tilde{\Lambda}_f^{W_h}}{\mathrm{d}\log\mu_i}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k} - \sum_f \tilde{\Lambda}_f^{W_h}\frac{\mathrm{d}^2\log\Lambda_f}{\mathrm{d}\log\mu_i\,\mathrm{d}\log\mu_k} - \frac{\mathrm{d}\,\tilde{\lambda}_k^{W_h}}{\mathrm{d}\log\mu_i}\Bigg).$$

At the efficient point, this simplifies to

$$\begin{aligned}\frac{\mathrm{d}^2\log W^{BS}}{\mathrm{d}\log\mu_k\,\mathrm{d}\log\mu_i} &= \sum_f \frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\left(\frac{\mathrm{d}\,\Lambda_f}{\mathrm{d}\log\mu_i} - \sum_h \overline{\chi_h^W}\frac{\mathrm{d}\,\tilde{\Lambda}_f^{W_h}}{\mathrm{d}\log\mu_i}\right)\\ &\quad + \frac{\mathrm{d}\,\lambda_k}{\mathrm{d}\log\mu_i} - \sum_h \overline{\chi_h^W}\frac{\mathrm{d}\,\tilde{\lambda}_k^{W_h}}{\mathrm{d}\log\mu_i} - \sum_{f,h}\Lambda_f\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\frac{\mathrm{d}\log\chi_h^W}{\mathrm{d}\log\mu_i}\\ &\quad - \lambda_k\frac{\mathrm{d}\log\chi_h^W}{\mathrm{d}\log\mu_i} - \lambda_k\delta_{ki} + \frac{\mathrm{d}\,\lambda_i}{\mathrm{d}\log\mu_k}.\end{aligned}$$

By Lemma 8, at the efficient point,

$$\frac{\mathrm{d}\,\lambda_j}{\mathrm{d}\log\mu_i} - \sum_h \overline{\chi}_h^W\frac{\mathrm{d}\,\tilde{\lambda}_j^{W_h}}{\mathrm{d}\log\mu_i} = \sum_h \frac{\mathrm{d}\,\chi_h^W}{\mathrm{d}\log\mu_i}\tilde{\lambda}_j^{W_h} - \lambda_i\left(\Psi_{ij} - \delta_{ij}\right).$$

Whence, we can further simplify the previous expression to

$$\begin{aligned}\frac{\mathrm{d}^2\log W^{BS}}{\mathrm{d}\log\mu_k\,\mathrm{d}\log\mu_i} &= \sum_f \frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\left(\sum_h \frac{\mathrm{d}\,\chi_h^W}{\mathrm{d}\log\mu_i}\tilde{\Lambda}_f^{W_h} - \lambda_i\Psi_{if}\right)\\ &\quad + \sum_h \frac{\mathrm{d}\,\chi_h}{\mathrm{d}\log\mu_i}\tilde{\lambda}_k^{W_h} - \lambda_i(\Psi_{ik} - \delta_{ik}) - \sum_{f,h}\Lambda_f\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}\\ &\quad - \frac{\lambda_k}{\mathrm{d}\log\chi_h}\mathrm{d}\log\mu_i - \lambda_k\delta_{ki} + \frac{\mathrm{d}\,\lambda_i}{\mathrm{d}\log\mu_k},\\ &= \sum_f \frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\left(\sum_h \frac{\mathrm{d}\,\chi_h}{\mathrm{d}\log\mu_i}\tilde{\Lambda}_f^{W_h} - \lambda_i\Psi_{if}\right)\\ &\quad + \sum_h \frac{\mathrm{d}\,\chi_h}{\mathrm{d}\log\mu_i}\tilde{\lambda}_k^{W_h} - \lambda_i\Psi_{ik} - \sum_{f,h}\Lambda_f\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}\\ &\quad - \frac{\lambda_k}{\mathrm{d}\log\chi_h}\mathrm{d}\log\mu_i + \frac{\mathrm{d}\,\lambda_i}{\mathrm{d}\log\mu_k},\end{aligned}$$

and using $\mathrm{d}\log\lambda_i = \mathrm{d}\log p_i + \mathrm{d}\log y_i$,

$$= \sum_f \frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\left(\sum_h \frac{\mathrm{d}\,\chi_h}{\mathrm{d}\log\mu_i}\tilde{\Lambda}_f^{W_h} - \lambda_i\Psi_{if}\right)$$

$$
\begin{aligned}
&+\sum_h \frac{\mathrm{d}\,\chi_h}{\mathrm{d}\log\mu_i}\tilde{\lambda}_k^{W_h} - \lambda_i \Psi_{ik} - \sum_{f,h} \Lambda_f \frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i} \\
&- \frac{\lambda_k}{\mathrm{d}\log\chi_h}\mathrm{d}\log\mu_i + \lambda_i \frac{\mathrm{d}\log p_i}{\mathrm{d}\log\mu_k} + \lambda_i \frac{\mathrm{d}\log y_i}{\mathrm{d}\log\mu_k}, \\
&= \sum_{f,h} \chi_h \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}\tilde{\Lambda}_f^{W_h}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k} - \lambda_i \sum_f \Psi_{if}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k} \\
&+ \sum_h \chi_h \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}\tilde{\lambda}_k^{W_h} - \lambda_i\Psi_{ik} - \sum_{f,h}\Lambda_f \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k} \\
&- \lambda_k \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i} + \lambda_i \frac{\mathrm{d}\log y_i}{\mathrm{d}\log\mu_k} \\
&+ \lambda_i \left(\sum_f \Psi_{if}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k} + \Psi_{ik}\right), \\
&= \sum_{f,h} \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\left(\chi_h\tilde{\Lambda}_f^{W_h} - \Lambda_f\right) \\
&+ \lambda_i \frac{\mathrm{d}\log y_i}{\mathrm{d}\log\mu_k} + \sum_h \chi_h \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}\tilde{\lambda}_k^{W_h} - \lambda_k \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}, \\
&= \lambda_i \frac{\mathrm{d}\log y_i}{\mathrm{d}\log\mu_k} + \sum_h \chi_h \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}\left(\tilde{\Lambda}_f^{W_h}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k} + \tilde{\lambda}_k^{W_h}\right) \\
&- \sum_{f,h}\frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\Lambda_f - \lambda_k \sum_h \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}, \\
&= \lambda_i \frac{\mathrm{d}\log y_i}{\mathrm{d}\log\mu_k} + \sum_h \chi_h \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}\frac{\mathrm{d}\log P_{cpi,h}}{\mathrm{d}\log\mu_k} \\
&- \left(\sum_f \frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\Lambda_f\right)\left(\sum_h \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}\right) - \lambda_k \sum_h \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}, \\
&= \lambda_i \frac{\mathrm{d}\log y_i}{\mathrm{d}\log\mu_k} + Cov_\chi\left(\frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}, \frac{\mathrm{d}\log P_{cpi,h}}{\mathrm{d}\log\mu_k}\right),
\end{aligned}
$$

since

$$
-\sum_f \frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\Lambda_f = -\sum_f \frac{\mathrm{d}\,\Lambda_f}{\mathrm{d}\log\mu_k} = \frac{\mathrm{d}\left(1-\sum_j \lambda_j(1-\frac{1}{\mu_j})\right)}{\mathrm{d}\log\mu_k} = -\lambda_k
$$

at the efficient point, and

$$
\sum_h \chi_h \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i} = 0.
$$

■

Proof of Theorem 6. From Theorem 4, we have

$$\mathcal{L} = -\frac{1}{2}\sum_l (d\log\mu_l)\lambda_l d\log y_l.$$

With the maintained normalization $PY = 1$, we also have

$$d\log y_l = d\log\lambda_l - d\log p_l,$$

$$d\log p_l = \sum_f \Psi_{lf} d\log\Lambda_f + \sum_k \Psi_{lk} d\log\mu_k,$$

where, from Theorem 3,

$$\begin{aligned} d\log\lambda_l = &\sum_k (\delta_{lk} - \frac{\lambda_k}{\lambda_l}\Psi_{kl}) d\log\mu_k - \sum_j \frac{\lambda_j}{\lambda_l}(\theta_j - 1) Cov_{\Omega^{(j)}}(\sum_k \Psi_{(k)} d\log\mu_k - \sum_g \Psi_{(g)} d\log\Lambda_g, \Psi_{(l)}) \\ &+ \frac{1}{\lambda_l}\sum_{g\in F^*}\sum_c \left(\lambda_l^{W_c} - \lambda_l\right)\Phi_{cg}\Lambda_g d\log\Lambda_g, \end{aligned}$$

and

$$\begin{aligned} d\log\Lambda_f = &-\sum_k \lambda_k \frac{\Psi_{kf}}{\Lambda_f} d\log\mu_k - \sum_j \lambda_j(\theta_j - 1) Cov_{\Omega^{(j)}}(\sum_k \Psi_{(k)} d\log\mu_k - \sum_g \Psi_{(g)} d\log\Lambda_g, \frac{\Psi_{(f)}}{\Lambda_f}) \\ &+ \frac{1}{\Lambda_f}\sum_{g\in F^*}\sum_c \left(\Lambda_i^{W_c} - \Lambda_f\right)\Phi_{cg}\Lambda_g d\log\Lambda_g. \end{aligned}$$

We will now use these expressions to replace in formula for the second-order loss function. We get

$$\begin{aligned} \mathcal{L} = &-\frac{1}{2}\sum_l\sum_k (\frac{\delta_{lk}}{\lambda_k} - \frac{\Psi_{kl}}{\lambda_l} - \frac{\Psi_{lk}}{\lambda_k})\lambda_k\lambda_l d\log\mu_k d\log\mu_l + \frac{1}{2}\sum_l \lambda_l d\log\mu_l \sum_f \Psi_{lf} d\log\Lambda_f \\ &+ \frac{1}{2}\sum_l\sum_j (d\log\mu_l)\lambda_j(\theta_j - 1) Cov_{\Omega^{(j)}}(\sum_k \Psi_{(k)} d\log\mu_k - \sum_g \Psi_{(g)} d\log\Lambda_g, \Psi_{(l)}) \\ &- \frac{1}{2}\sum_l d\log\mu_l \left(\sum_g\sum_c \left(\lambda_l^{W_c} - \lambda_l\right)\Phi_{cg}\Lambda_g d\log\Lambda_g\right) \end{aligned}$$

$$\mathcal{L} = -\frac{1}{2}\sum_l\sum_k (\frac{\delta_{lk}}{\lambda_k} - \frac{\Psi_{kl}}{\lambda_l} - \frac{\Psi_{lk}}{\lambda_k})\lambda_k\lambda_l d\log\mu_k d\log\mu_l + \frac{1}{2}\sum_l \lambda_l d\log\mu_l \sum_f \Psi_{lf} d\log\Lambda_f$$

$$+\frac{1}{2}\sum_l\sum_j(d\log\mu_l)\lambda_j(\theta_j-1)Cov_{\Omega^{(j)}}(\sum_k\Psi_{(k)}d\log\mu_k-\sum_g\Psi_{(g)}d\log\Lambda_g,\Psi_{(l)})$$
$$-\frac{1}{2}\sum_l\left(\sum_c\left(\lambda_l^{W_c}-\lambda_l\right)\chi_c d\log\chi_c\right)d\log\mu_l$$

We can rewrite this expression as

$$\mathcal{L}=\mathcal{L}_I+\mathcal{L}_X+\mathcal{L}_H$$

where

$$\mathcal{L}_I=\frac{1}{2}\sum_k\sum_l[\frac{\Psi_{kl}-\delta_{kl}}{\lambda_l}+\frac{\Psi_{lk}-\delta_{lk}}{\lambda_k}+\frac{\delta_{kl}}{\lambda_l}-1]\lambda_k\lambda_l d\log\mu_k d\log\mu_l$$
$$+\frac{1}{2}\sum_k\sum_l\sum_j d\log\mu_k d\log\mu_l\lambda_j(\theta_j-1)Cov_{\Omega^{(j)}}(\Psi_{(k)},\Psi_{(l)}),$$

$$\mathcal{L}_X=\frac{1}{2}\sum_l\sum_f(\frac{\Psi_{lf}}{\Lambda_f}-1)\lambda_l\Lambda_f d\log\mu_l d\log\Lambda_f$$
$$-\frac{1}{2}\sum_l\sum_g d\log\mu_l d\log\Lambda_g\sum_j\lambda_j(\theta_j-1)Cov_{\Omega^{(j)}}(\Psi_{(g)},\Psi_{(l)}),$$

$$\mathcal{L}_H=-\frac{1}{2}\sum_l\left(\sum_c\left(\lambda_l^{W_c}-\lambda_l\right)\chi_c d\log\chi_c\right)d\log\mu_l,$$

where $d\log\Lambda$ is given by the usual expression.[7] Finally, using Lemma 11, we can write

$$\mathcal{L}_{\mathcal{I}}=\frac{1}{2}\sum_l\sum_k(d\log\mu_l)(d\log\mu_k)\sum_j\lambda_j\theta_j Cov_{\Omega^{(j)}}(\Psi_{(k)},\Psi_{(l)}).$$

and

$$\mathcal{L}_X=-\frac{1}{2}\sum_l\sum_g d\log\mu_l d\log\Lambda_g\sum_j\lambda_j\theta_j Cov_{\Omega^{(j)}}(\Psi_{(g)},\Psi_{(l)}).$$

[7]We have used the intermediate step

$$\mathcal{L}_X=\frac{1}{2}\sum_l\sum_k\lambda_k\lambda_l d\log\mu_k d\log\mu_l+\frac{1}{2}\sum_l\sum_f d\log\mu_l d\log\Lambda_f\lambda_l\Psi_{lf}$$
$$-\frac{1}{2}\sum_l\sum_g d\log\mu_l d\log\Lambda_g\sum_j\lambda_j(\theta_j-1)Cov_{\Omega^{(j)}}(\Psi_{(g)},\Psi_{(l)}).$$

$\blacksquare$

Lemma 10. *The following identity holds*

$$\sum_j \lambda_j \left(\tilde{\Psi}_{jk} \Psi_{jl} - \sum_m \Omega_{jm} \tilde{\Psi}_{mk} \Psi_{ml} \right) = \tilde{\lambda}_k \lambda_l.$$

Proof. Write Ω so that it contains all the producers, all the households, and all the factors *as well as* a new row (indexed by 0) where $\Omega_{0i} = \chi_i$ if $i \in C$ and 0 otherwise. then, letting e_0 be the standard basis vector corresponding to the 0th row, we can write

$$\lambda' = e_0' + \lambda' \Omega,$$

or equivalently

$$\lambda'(I - \Omega) = e_0'.$$

Let X^{kl} be the vector where $X^{kl}_m = \tilde{\Psi}_{mk} \Psi_{ml}$. Then

$$\sum_j \lambda_j \left(\tilde{\Psi}_{jk} \Psi_{jl} - \sum_m \Omega_{jm} \tilde{\Psi}_{mk} \Psi_{ml} \right) = \lambda'(I - \Omega) X^{kl},$$

$$= e_0'(I - \Omega)^{-1}(I - \Omega) X^{kl}, \quad = e_0' X^{kl} = \tilde{\Psi}_{0k} \Psi_{0l} = \tilde{\lambda}_k \lambda_l.$$

$\blacksquare$

Lemma 11. *The following identity holds*

$$\sum_j \lambda_j \mu_j^{-1} Cov_{\tilde{\Omega}^{(j)}}(\tilde{\Psi}_{(k)}, \Psi_{(l)}) = \lambda_l \lambda_k [\frac{\tilde{\Psi}_{lk} - \delta_{lk}}{\lambda_k} + \frac{\Psi_{kl} - \delta_{kl}}{\lambda_l} + \frac{\delta_{lk}}{\lambda_k} - \frac{\tilde{\lambda}_k}{\lambda_k}].$$

Proof. We have

$$\sum_j \lambda_j \mu_j^{-1} Cov_{\tilde{\Omega}^{(j)}}(\tilde{\Psi}_{(k)}, \Psi_{(l)}) =$$

$$\sum_j \lambda_j \mu_j^{-1} \left[\sum_m \tilde{\Omega}_{jm} \tilde{\Psi}_{mk} \Psi_{ml} - \left(\sum_m \tilde{\Omega}_{jm} \tilde{\Psi}_{mk} \right) \left(\sum_m \tilde{\Omega}_{jm} \Psi_{ml} \right) \right],$$

or

$$\sum_j \lambda_j \mu_j^{-1} Cov_{\tilde{\Omega}^{(j)}}(\tilde{\Psi}_{(k)}, \Psi_{(l)}) =$$

$$\sum_j \lambda_j \sum_m \Omega_{jm} \tilde{\Psi}_{mk} \Psi_{ml} - \sum_j \lambda_j \mu_j^{-1} \left(\sum_m \tilde{\Omega}_{jm} \tilde{\Psi}_{mk} \right) \left(\sum_m \tilde{\Omega}_{jm} \Psi_{ml} \right),$$

or

$$\sum_j \lambda_j \mu_j^{-1} Cov_{\tilde{\Omega}^{(j)}}(\tilde{\Psi}_{(k)}, \Psi_{(l)}) =$$
$$\sum_j \lambda_j \sum_m \Omega_{jm} \tilde{\Psi}_{mk} \Psi_{ml} - \sum_j \lambda_j \tilde{\Psi}_{jk} \Psi_{jl}$$
$$+ \sum_j \lambda_j \tilde{\Psi}_{jk} \Psi_{jl} - \sum_j \lambda_j \mu_j^{-1} \left(\sum_m \tilde{\Omega}_{jm} \tilde{\Psi}_{mk} \right) \left(\sum_m \tilde{\Omega}_{jm} \Psi_{ml} \right),$$

or using, Lemma 10

$$\sum_j \lambda_j \mu_j^{-1} Cov_{\tilde{\Omega}^{(j)}}(\tilde{\Psi}_{(k)}, \Psi_{(l)}) = -\tilde{\lambda}_k \lambda_l + \sum_j \lambda_j \tilde{\Psi}_{jk} \Psi_{jl} - \sum_j \lambda_j \left(\tilde{\Psi}_{jk} - \delta_{jk} \right) \left(\Psi_{jl} - \delta_{jl} \right),$$

and finally

$$\sum_j \lambda_j \mu_j^{-1} Cov_{\tilde{\Omega}^{(j)}}(\tilde{\Psi}_{(k)}, \Psi_{(l)}) = \lambda_l \lambda_k [\frac{\tilde{\Psi}_{lk} - \delta_{lk}}{\lambda_k} + \frac{\Psi_{kl} - \delta_{kl}}{\lambda_l} + \frac{\delta_{lk}}{\lambda_k} - \frac{\tilde{\lambda}_k}{\lambda_k}].$$

■

Proposition 3 (Structural Output Loss)**.** *Starting at an efficient equilibrium in response to the introduction of small tariffs or other distortions,*

$$\Delta \log Y \approx -\frac{1}{2} \sum_{l \in N} \sum_{k \in N} \Delta \log \mu_k \Delta \log \mu_l \sum_{j \in N} \lambda_j \theta_j Cov_{\Omega^{(j)}}(\Psi_{(k)}, \Psi_{(l)})$$
$$- \frac{1}{2} \sum_{l \in N} \sum_{g \in F} \Delta \log \Lambda_g \Delta \log \mu_l \sum_{j \in N} \lambda_j \theta_j Cov_{\Omega^{(j)}}(\Psi_{(g)}, \Psi_{(l)})$$
$$+ \frac{1}{2} \sum_{l \in N} \sum_{c \in C} \chi_c^W \Delta \log \chi_c^W \Delta \log \mu_l (\lambda_l^{W_c} - \lambda_l).$$

Proof. The proof follows along the same lines as Theorem 6. ■

L Growth Accounting Results

Table 2: Decomposition of real GNE growth

	GNE	GDP	ToT	Technology	Factoral ToT	Transfers
AUS	0.665	0.526	0.134	0.619	0.041	0.006
AUT	0.213	0.315	0.000	0.402	-0.087	-0.102
BEL	0.285	0.252	0.038	0.431	-0.142	-0.004
BGR	0.322	-0.217	0.354	-0.145	0.282	0.185
BRA	0.549	0.532	0.049	0.538	0.043	-0.032
CAN	0.630	0.525	0.110	0.581	0.055	-0.005
CHN	1.780	1.810	0.159	1.583	0.386	-0.188
CYP	0.322	0.275	-0.046	0.261	-0.033	0.093
CZE	0.413	0.283	0.295	0.412	0.166	-0.166
DEU	0.160	0.306	-0.013	0.428	-0.135	-0.132
DNK	0.239	0.199	0.095	0.318	-0.024	-0.056
ESP	0.330	0.280	0.003	0.346	-0.063	0.047
EST	0.793	0.125	0.661	0.351	0.435	0.008
FIN	0.347	0.432	-0.121	0.386	-0.075	0.037
FRA	0.317	0.358	-0.079	0.374	-0.095	0.038
GBR	0.437	0.358	0.058	0.465	-0.049	0.021
GRC	0.165	0.130	-0.027	0.110	-0.006	0.062
HUN	0.326	0.278	0.141	0.308	0.111	-0.092
IDN	0.633	0.660	-0.006	0.684	-0.030	-0.020
IND	1.236	1.264	-0.043	1.169	0.053	0.015
IRL	0.503	0.575	0.290	0.482	0.383	-0.361
ITA	0.072	-0.008	0.082	0.182	-0.108	-0.002
JPN	0.034	0.104	-0.102	0.187	-0.185	0.032
KOR	0.590	0.834	-0.149	0.739	-0.054	-0.094
LTU	0.739	0.515	0.187	0.423	0.278	0.038
LUX	0.605	0.162	0.979	0.581	0.561	-0.537
LVA	0.728	0.095	0.404	0.263	0.235	0.230
MEX	0.640	0.526	0.090	0.537	0.079	0.023
MLT	0.432	0.464	0.124	0.345	0.243	-0.156
NLD	0.249	0.374	0.009	0.495	-0.112	-0.134
POL	0.746	0.779	-0.039	0.638	0.101	0.006
PRT	0.096	0.041	0.040	0.131	-0.051	0.016
ROU	0.698	0.397	0.189	0.277	0.308	0.112
RUS	0.721	0.583	0.315	0.632	0.267	-0.178
SVK	0.690	0.557	0.196	0.403	0.349	-0.063
SVN	0.339	0.391	0.015	0.398	0.009	-0.067
SWE	0.360	0.413	-0.014	0.443	-0.045	-0.039
TUR	0.849	0.986	-0.232	0.794	-0.040	0.096
TWN	0.502	1.066	-0.410	0.727	-0.070	-0.155
USA	0.431	0.391	-0.007	0.431	-0.046	0.047
ROW	0.753	0.655	0.084	0.639	0.101	0.014

The sample is 1996-2014. Each row decomposes the cumulative log change in real GNE for each country. The first decomposition follows (7). Columns 2, 3 and 6 sum to column 1. The second decomposition follows (6). Columns 4, 5, and 6 sum to column 1.

M Additional Examples

M.1 Example of an economy in standard form

We use a two-country example to show how to map a specific nested-CES model into *standard-form* required by Theorem 3. Suppose there are n industries at home and foreign. The utility function of home and foreign consumers is

$$W = \prod_{i=1}^{n} (x_{0i})^{\Omega_{0i}}, \qquad W_* = \prod_{i=1}^{n} (x_{0i}^*)^{\Omega_{0i}},$$

where x_{0i} and x_{0i}^* are home and foreign consumption of goods from industry i. The production function of industry i (at home or foreign) is a Cobb-Douglas aggregate of intermediates and the local factor

$$y_i = L_{ij}^{\Omega_{iL}} \prod_{i=1}^{n} x_{ij}^{\Omega_{ij}}.$$

Suppose that the intermediate good x_{ij} is a CES combination of domestic and foreign varieties of j, with initial home share Ω_j and foreign share $\Omega_j^* = 1 - \Omega_j$, and elasticity of substitution $\varepsilon_j + 1$. Since the market share of home and foreign in industry j does not vary by consumer i, this means there is no home-bias.

In standard-form, this economy has $N = 3n$ producers: the first n are industries at home, the second n are industries in foreign, and the last n are CES aggregates of domestic and foreign varieties that every other industry buys. The HAIO matrix for this economy, in standard-form, is $(2 + 3n + 2) \times (2 + 3n + 2)$:

$$\Omega = \left[\begin{array}{c|c|c|c|c c}
\mathbf{0} & \mathbf{0} & \mathbf{0} & \left[\Omega_{0i}\right]_{i=1}^{n} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \left[\Omega_{0i}\right]_{i=1}^{n} & \mathbf{0} & \mathbf{0} \\
\hline
\mathbf{0} & \mathbf{0} & \mathbf{0} & \left[\Omega_{ij}\right]_{i,j=1}^{n} & \left[\Omega_{iL}\right]_{i=1}^{n} & \mathbf{0} \\
\hline
\mathbf{0} & \mathbf{0} & \mathbf{0} & \left[\Omega_{ij}\right]_{i,j=1}^{n} & \mathbf{0} & \left[\Omega_{iL}\right]_{i=1}^{n} \\
\hline
\mathbf{0} & \begin{matrix} \Omega_1 & \cdots & 0 \\ & \ddots & \\ 0 & & \Omega_n \end{matrix} & \begin{matrix} \Omega_1^* & \cdots & 0 \\ & \ddots & \\ 0 & & \Omega_n^* \end{matrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\hline
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}
\end{array}\right].$$

The first two rows and columns correspond to the households, the next $2n$ rows and columns correspond to home industries and foreign industries respectively. The next n rows and columns correspond to bundles of home and foreign varieties. The last two rows and columns correspond to the home and foreign factor. The vector elasticities of substitution θ for this economy is a vector with $2+3n$ elements $\theta = (1, \cdots, 1, \varepsilon_1 + 1, \cdots, \varepsilon_n + 1)$, where ε_i is the trade elasticity in industry i.

Now that we have written this economy in standard-form, we can use Theorem 3 to study the change in home's share of income following a productivity shock $d\log A_j$ to some *domestic* producer j:

$$\frac{d\log \Lambda_L}{d\log A_j} = \frac{\lambda_j}{\Lambda_L} \frac{\varepsilon_j \Omega_j^* \Omega_{jL}}{1 + \sum_i \varepsilon_i \frac{\lambda_i \Omega_{iL}}{\Lambda_L} \frac{\Omega_{iL}}{1-\Lambda_L} \Omega_i^*} \geq 0,$$

which is positive as long as domestic and foreign varieties are substitutes $\varepsilon_j > 0$ for every j. The numerator captures the fact that a shock to j will increase demand for the home factor if j uses the home factor $\Omega_{jL} > 0$. The denominator captures the fact that an increase in the price of the home factor attenuates the increase in demand for the home factor by bidding up the price of home goods.

The positive productivity shock to j will therefore shrink the market share of every other domestic producer, a phenomenon known as Dutch disease. To see this, apply Theorem 3 to some domestic producer $i \neq j$ to get

$$\frac{d\log \lambda_i}{d\log A_j} = -\varepsilon_i \Omega_i^* \frac{\Omega_{iL}}{1-\Lambda_L} \frac{d\log \Lambda_L}{d\log A_j} < 0.$$

In words, the shock to j boosts the price of the home factor, which makes i less competitive in the world market if i relies on the home factor $\Omega_{iL} > 0$. Hence, if $\varepsilon_j > 0$ for every j, a domestic productivity shock to one sector will cause Dutch disease and shrink the market share of other domestic producers by bidding up home wages.

M.2 More details on Example IV from Section 6

First, the forward propagation equations (8) from Theorem 3 imply that the change in the price of each good is

$$d\log p = \sum_{k \in N} \Psi_{(k)} d\log \mu_k + \frac{\Psi_{(L)}}{\Lambda_L} d\Lambda_L - \frac{(1-\Psi_{(L)})}{1-\Lambda_L} \left[d\Lambda_L + \sum_i \lambda_i d\log \mu_i \right].$$

The first-term captures the direct effect of the tariff on the price of each good, the second term captures the effect of the change in the wage of domestic workers, and the last term captures the effect of changes in the foreign wage. Here, we use the fact that the change in the foreign wage relative to world GDP is the negative of the change in the home wage and the tax revenues collected (the expression in square brackets).

Substituting the expression for prices into the backward propagation equations from Theorem 3 yields the following expression for the home factor's change in aggregate income:

$$d\Lambda_L = \frac{-d\log\mu_L + \sum_{k\in N}\lambda_k(1-\theta_k)Cov_{\Omega^{(k)}}(\Psi_{(L)}, \Psi_{(M)}d\log\mu + \Psi_{(L)}\frac{d\Lambda_R}{1-\Lambda_L}) + (\Lambda_L^{W_L} - \Lambda_L^{W_{L*}})d\Lambda_R}{1 - \frac{1}{\Lambda_L(1-\Lambda_L)}\sum_{k\in N}\lambda_k(1-\theta_k)Var_{\Omega^{(k)}}(\Psi_{(L)}) - (\Lambda_L^{W} - \Lambda_L^{W_*})}, \tag{24}$$

where $d\log\mu_L = \sum_k \lambda_k \Psi_{kL} d\log\mu_k$ and $\Psi_{(M)}d\log\mu = \sum_{k\in N}\Psi_{(k)}d\log\mu_k$. The tariff revenues are $d\Lambda_R = \sum_k \lambda_k d\log\mu_k$. Each term in (24) is intuitive: the numerator is the effect of the tax in partial equilibrium, holding fixed factor prices in terms of world GDP. The denominator is the general equilibrium effect capturing the endogenous substitution and income redistribution effects triggered by changes in factor prices — that is, the fixed point depicted in Figure 1.

To understand the intuition, consider the numerator, which consists of three effects. The first summand in the numerator is the direct incidence of the tax on the home labor, taking into account supply chains. The second term, involving the covariance, is how the tax causes substitution by changing relative prices of goods, and the covariance captures whether or not goods whose relative prices rise tend to be reliant on home labor. The final term in the numerator captures the fact that the tariff revenues, by redistributing income between home and foreign, change demand for the domestic factor. The denominator then accounts for the fact that the partial equilibrium change in factor prices result in additional rounds of expenditure-switching due to substitution and income redistribution.

From home's perspective, the ideal tariff, which raises home wages relative to foreign wages, is one which is imposed on goods that do not directly or indirectly use the domestic factor. For such goods, $d\log\mu_L = 0$. Furthermore, if substitution elasticities are greater than one, $\theta_k \geq 1$, then the ideal tariff should be levied on goods which negatively correlate with domestic factor usage, in which case $Cov_{\Omega^{(k)}}(\Psi_{(L)}, \Psi_{(M)}d\log\mu) < 0$. In other words, if a good is heavily exposed to the tax, then it should also be heavily exposed to foreign (rather than domestic) labor.