Growth, Automation and the Long Run Share of Labor

SUPPLEMENTARY APPENDIX

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Here, we provide omitted proofs of some results from the main text. All numbered references for figures, equations, lemmas, etc. refer to the main text. References that start with "a" refer to corresponding objects in this Appendix.

Proof of Theorem 2:

We start by showing that prices must be bounded away from zero:

LEMMA **A.1.** *Let* $\mathbf{p}(t)$ *be a corresponding sequence of equilibrium prices for any equilibrium with* $Y(t) \to \infty$. *Then for each good* i, $\liminf_t p_i(t) > 0$.

Proof. Since input prices are always positive, $\mathbf{p}(t) \gg 0$ for every t, so demands are well-defined. If the Lemma is false, then by Well-Behaved Preferences, there is a subsequence of dates (retain original index t) and a good i with $p_i(t) \to 0$ and $\lim_t d_{im}(\mathbf{p}(t)) > 0$ for every m.

Because $Y(t) \to \infty$, it is easy to see that expenditures on final goods for some type m must also grow to infinity. Therefore total demand for good i, which is bounded below by $d_{im}(\mathbf{p}(t))z_m(t)$, must grow to infinity. Moreover, because the price of this good converges to zero, the labor input price $\lambda_i(t) \to 0$. It follows that the labor-capital ratio in this sector must be bounded away from 0 in t (it goes to infinity, in fact). Because the amount of human labor is finite, robot use in this sector goes to infinity. In particular the *production* of robots goes to infinity. By (17), labor input prices in the robot sector are bounded above, so the labor input in the robot sector goes to infinity. But then the finiteness of the human labor endowment in the economy implies that the robot sector must eventually be automated, and $p_r(t) = p_r^*$ for all large t. Returning to sector i, and remembering that i is automated for large enough t, we now have a contradiction to $p_i(t) \to 0$, because prices of both inputs (capital and labor) are constant for all large t. ■

LEMMA **A.2.** *Let S be the set of all infinite-dimensional nonnegative vectors $\mathbf{s} \equiv (s_1, s_2, \ldots)$, with components in $[0,1]$ and with $\sum_{j=1}^{\infty} s_j = 1$. Let $\mathbf{s}(t)$ be a sequence in S indexed by t, and suppose that there is $\hat{\mathbf{s}} \in S$ such that $\mathbf{s}(t)$ converges pointwise to $\hat{\mathbf{s}} = (\hat{s}_j)$. Let $\mathbf{\Psi}(t)$ be a corresponding convergent sequence with components $(\Psi_1(t), \Psi_2(t), \ldots)$, where $\Psi_j(t) \in [0,1]$ for every j and t, with $\Psi_j(t) \to 0$ as $t \to \infty$ for every j with $\hat{s}_j > 0$. Then*

$$\lim_{t\to\infty} \sum_{j=1}^{\infty} \Psi_j(t)\hat{s}_j(t) = 0. \tag{a.1}$$

Proof. For any n, let J be some positive integer such that for $\hat{\mathbf{s}}$ in the statement of the lemma, $\sum_{j=1}^{J} \hat{s}_j \geq 1 - (1/2)^{n+2}$. Then there exists $T_1(n)$ such that along the sequence $\{\mathbf{s}(t)\}$,

$$\sum_{i=1}^{J} s_i(t) \geq 1 - (1/2)^{n+2} - (1/2)^{n+2} = 1 - (1/2)^{n+1}$$

for $t \geq T_1(n)$, using pointwise convergence on the finite set $\{1, \ldots, J\}$. Because $\Psi_i(t) \in [0,1]$ for all i and t and $\sum_j s_j(t) = 1$ for every t, it follows that for $t \geq T_1(n)$,

$$\sum_{j=J+1}^{\infty} \Psi_j(t)s_j(t) \leq \sum_{j=J+1}^{\infty} s_j(t) < (1/2)^{n+1}. \tag{a.2}$$

Because $\Psi_j(t) \to 0$ as $t \to \infty$ for every j with $\hat{s}_j > 0$, we know that $\Psi_j(t)s_j(t) \to 0$. Therefore there exists $T(n) \geq T_1(n)$ so that in addition to (a.2),

$$\sum_{i=1}^{J} \Psi_i(t)s_i(t) \leq (1/2)^{n+1} \tag{a.3}$$

for $t \geq T(n)$. Combining (a.2) and (a.3), we must conclude that

$$\sum_{j=1}^{\infty} \Psi_j(t)s_j(t) < (1/2)^n.$$

for $t \geq T(n)$. Because n can be made arbitrarily large, the proof is complete. ■

Now return to the main proof.

Part (i). Call a sector *consequential* if its output goes to ∞ along a subsequence. We claim that there is a consequential final-goods sector. By the singularity condition and Proposition 1, all equilibrium prices are bounded above:

$$p_j(t) = c_j(1, \lambda_j(t)) \le c_j(1, \nu_j^{-1} p_r(t)) \le c_j(1, \nu_j^{-1} p_r^*). \tag{a.4}$$

Using a diagonal argument, extract a convergent subsequence of equilibrium prices that converges pointwise (but retain original index t) to some price vector $\mathbf{p}^*$. By Lemma A.1, $\mathbf{p}^* \gg 0$, and so, applying Well-Behaved Preferences, we have $\mathbf{d}_m(\mathbf{p}(t))$ converging pointwise to $\mathbf{d}_m(\mathbf{p}^*)$. It is easy to see that per-capita consumption expenditures on final goods must go to ∞. Pick any type m for whom $z_m(t) \to \infty$. Obviously, $d_{im}(\mathbf{p}^*) > 0$ for some good i. Combining all this information, we must conclude that for at least one final good sector, demand must go to infinity along this subsequence. The rest of the proof is identical to that of part (i) in Theorem 1.

Part (ii). Notice that

$$\Psi_j(t) = \frac{h_j(t)}{\nu_j r_j(t) + h_j(t)} = \frac{w_j(t)h_j(t)}{p_r(t)r_j(t) + w_j(t)h_j(t)}$$

for every sector j and every date t.* If $\sigma(t)$ denotes the share of human labor in national income, it follows that

$$\sigma(t) = \frac{\sum_j w_j(t)h_j(t)}{Y(t)} = \frac{\sum_j \Psi_j(t)[p_r(t)r_j(t) + w_j(t)h_j(t)]}{Y(t)} \le \frac{\sum_j \Psi_j(t)p_j(t)y_j(t)}{Y(t)}$$

$$= \left[\frac{Z(t)}{Y(t)}\right]\left[\frac{\sum_{i=1}^{\infty} \Psi_i(t)p_i(t)y_i(t)}{Z(t)}\right] + \frac{\sum_{j=e,r,k} \Psi_j(t)p_j(t)y_j(t)}{Y(t)} \tag{a.5}$$

at every date t, where $Z(t)$ stands for economy-wide per-capita current expenditures on final goods. Because there are finitely many homothetic preference groups indexed by m, we can write for every final good i and date t:

$$\frac{p_i(t)y_i(t)}{Z(t)} = \sum_m \zeta_m(t)s_{mi}(t) \tag{a.6}$$

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*This is trivial when either $r_j(t)$ or $h_j(t)$ equals zero; but when both are positive, $p_r(t) = \nu_j w_j(t)$.

where $\zeta_m(t)$ is the expenditure share of preference type m in total expenditure, and $s_{mi}(t)$ is the corresponding expenditure share of good i for preference group m. Because each of the terms $\zeta_m(t)$ are bounded above by 1, we may combine (a.5) and (a.6) to write:

$$\text{(a.7)}\quad \sigma(t) \leq Q(t) \equiv \frac{Z(t)}{Y(t)} \sum_{i=1}^{\infty} \Psi_i(t) \left[\sum_m \zeta_m(t) s_{mi}(t) \right] + \frac{\sum_{j=e,r,k} \Psi_j(t) p_j(t) y_j(t)}{Y(t)}$$

for all t. We will show that the right hand side of (a.7) — defined to be $Q(t)$ — converges to 0 as $t \to \infty$. To this end, pick any subsequence of dates (but retain original notation) so that $Q(t)$ converges. Exploiting the fact that the number of sectors is countable, using a diagonal argument to extract a further subsequence (again retain notation) so that *each* of the sequences $\Psi_j(t)$, $\zeta_m(t)$, $s_{mi}(t)$, $p_j(t)$, $Z(t)/Y(t)$ and $[p_j(t)y_j(t)]/Y(t)$ also converge. The last term in $Q(t)$ pertains only to three sectors: e, r and k. For any of these sectors, call it j, $\Psi_j(t) \to 0$ along any subsequence for which j is consequential, and on any other subsequence $p_j(t)y_j(t)$ must be bounded. Putting these observations together with $Y(t) \to \infty$, we must conclude that the last term in $Q(t)$ converges to 0.

Now for the first term. If $Z(t)/Y(t) \to 0$, we are done, so assume in what follows that $Z(t)/Y(t)$ has a strictly positive limit. Let M be the set of all indices for which $\lim_t \zeta_m(t) > 0$ for the subsequence under consideration. Then, using the fact that the interchange of a finite and infinite sum is always valid, we have

$$\begin{aligned} \sum_{i=1}^{\infty} \Psi_i(t) \left[\sum_m \zeta_m(t) s_{mi}(t) \right] &= \sum_m \zeta_m(t) \left[\sum_{i=1}^{\infty} \Psi_i(t) s_{mi}(t) \right] \\ &= \sum_{m \in M} \zeta_m(t) \left[\sum_{i=1}^{\infty} \Psi_i(t) s_{mi}(t) \right] + \sum_{m \notin M} \zeta_m(t) \left[\sum_{i=1}^{\infty} \Psi_i(t) s_{mi}(t) \right] \\ \text{(a.8)} \qquad &\leq \sum_m \left[\sum_{i=1}^{\infty} \Psi_i(t) s_{mi}(t) \right] + \sum_{m \notin M} \zeta_m(t) \left[\sum_{i=1}^{\infty} \Psi_i(t) s_{mi}(t) \right]. \end{aligned}$$

Because $\zeta_m(t) \to 0$ for all $m \notin M$, the second term on the right hand side of this equation converges to 0. It remains to show that same is true of the first term. It will suffice to show that for each $m \in M$,

$$\text{(a.9)}\qquad \sum_{i=1}^{\infty} \Psi_i(t) s_{mi}(t) \to 0$$

as $t \to \infty$ along our chosen subsequence. Because $\lim_t \zeta_m(t) > 0$ for $m \in M$, and given that $Z(t) \to \infty$ (after all, $Z(t)/Y(t)$ has a positive limit and $Y(t) \to \infty$) it follows that expenditures diverge to infinity for a positive measure of individuals of type m. Let $Z_m(t)$ be the aggregate income of type m and $x_{mi}(t)$ the aggregate demand for good i by this type. Homotheticity guarantees that

(a.10)
$$\hat{s}_{mi} \equiv \lim_t s_{mi}(t) = \lim_t \frac{p_i(t)x_{mi}(t)}{Y_m(t)} = \lim_t \frac{p_i(t)d_i^m(\mathbf{p}(t))Y_m(t)}{Y_m(t)} = \lim_t p_i(t)d_i^m(\mathbf{p}(t)),$$

By Lemma A.1 and Well-Behaved Preferences, it follows that $\hat{s}_{mi}$ forms a "bonafide share vector" with $\sum_i \hat{s}_{mi} = 1$. So the conditions in Lemma A.2 are satisfied (ignore index m). Therefore this Lemma implies (a.9), and the income share of human labor must converge to zero. Because (27) holds unchanged, the income share of capital converges to 1.

Part (iii). Let $\{\mathbf{p}(t)\}$ be the sequence of equilibrium prices. Given the bound (a.4), we can use a diagonal argument to extract a convergent subsequence $\{\mathbf{p}(t_s)\}$ which converges pointwise to some price sequence $\mathbf{p}^*$ as $s \to \infty$. By Lemma A.1, $\mathbf{p}^* \gg 0$, and so applying Well-Behaved Preferences, we see that for each type m, $\mathbf{d}_m(\mathbf{p}(t)) \to \mathbf{d}_m(\mathbf{p}^*)$ pointwise along this subsequence. By Positive Demand, $d_{mi}(\mathbf{p}^*) > 0$ for an infinite number of indices i. Putting all these arguments together and invoking homotheticity, we see that an infinite number of final goods sectors must be consequential.

We now claim that for any number W, however large, there exists a time T such that for all dates $t \geq T$, $w_j(t) \geq W$ for every sector j for which $h_j(t) > 0$. For suppose that this claim is false; then there exists some sector q and a subsequence of dates (retain original notation t) such that $\sup_t w_q(t) \equiv W_q < \infty$, but $h_q(t) > 0$. Next, pick a *consequential* final goods sector ℓ with the property that

$$\text{(a.11)} \qquad \nu_\ell^{-1} p_r^* > W_q + \bar{p}_e \sup_{j'} |E_{j'\ell} - E_{j'q}|,$$

where $\bar{p}_e$ is the upper bound on education prices given by (a.4). The existence of such a sector is guaranteed, first, by recalling that there is an infinite number of consequential sectors, then invoking Increasing Protection ($\nu_i \to 0$), and finally by Bounded Education Costs, which guarantees that the supremum in (a.11) is finite. Because sector ℓ is consequential, it is automated after some finite date so there exists T' such that for all $t \geq T'$ along the subsequence identified above, we have $w_\ell(t) \geq \nu_\ell^{-1} p_r^*$. Combining this information with (a.11), we must conclude that for every sector j, and for $t \geq T'$,

$$w_\ell(t) \geq \nu_\ell^{-1} p_r^* > W_q + \bar{p}_e \sup_{j'} |E_{j'\ell} - E_{j'q}| \geq w_q(t) + p_e(t)[E_{j\ell} - E_{jq}],$$

so that

$$w_\ell(t) - p_e(t) E_{j\ell} > w_q(t) - p_e(t) E_{jq}.$$

But this inequality proves that while sector ℓ might lie along a least-cost educational path, no individual can ever want to enter sector q at large dates, which implies that $h_q(t') = 0$ for all large enough t' along the subsequence, a contradiction. ■

Proof of Theorem 3. Consider the following infinite-horizon problem and case (i) (in case (ii) it is evident that the proof can be modified with γ denoting a lower bound on the return to capital). Suppose that an individual has a constant discount factor β and a (possibly time-varying) sequence of smooth, concave one-period utility indicators $u_t(z)$, each defined on a single consumption good z with unbounded steepness at zero. Suppose that the individual has F_0 units of a financial asset at date 0, receives a (fully anticipated) sequence of strictly positive incomes $\{y(t)\}$, and faces a constant return factor γ on financial holdings. She chooses a sequence $\{z(t), F(t)\}$ of consumptions and financial assets to maximize lifetime utility

$$\sum_{t=0}^{\infty} \beta^t u_t(z(t))$$

subject to the constraint that for every $t \geq 0$, $F(t) \geq B$ (a credit limit), $\liminf F(t) \geq 0$, and

$$y(t) + F(t) = z(t) + [F(t+1)/\gamma].$$

Assume that an optimum is well-defined, and assume $\beta\gamma > 1$. Then we claim that

$$\text{(a.12)} \qquad \lim_{t \to \infty} u_t'(z(t)) = 0.$$

The proof follows on observing that the Euler equation holds (with an appropriate inequality if there is a borrowing constraint):

$$u_t'(z(t)) \geq \beta\gamma u_{t+1}'(z(t+1))$$

for all t.[†] Because $\beta\gamma > 1$ by assumption, it follows that $u_t'(z(t))$ must decline (geometrically in fact) to zero.

We can transplant this problem easily to our setting, provided we view incomes here as wages net of education costs. Identifying $u_t(z)$ with $v_m(z, \mathbf{p}(t))$ for each individual m, it follows that in any equilibrium, if $\beta_m\gamma > 1$ (where γ is defined in (29)), we have

(a.13) $$\lim_{t\to 0} v_m'(z_m(t), \mathbf{p}(t)) = 0.$$

But v_m is strictly increasing and concave for every $\mathbf{p}$. Moreover, every final goods price is bounded above; see (a.4), Therefore (a.13) can *only* hold if $z_m(t) \to \infty$ as $t \to \infty$, and by our assumption, this must occur for a positive measure of individuals. With a bounded credit limit on the rest, it is easy to conclude that per-capita income $Y(t)$ as defined in (26) must go to infinity. ■

Proof of Lemma 1. Part (a). This follows immediately on inspecting equation (34).

Part (b). Suppose, on the contrary, that for some pair of sectors i and j, $R(t) \equiv \pi_i(t)/\pi_j(t) \to \infty$ along a subsequence of dates. We first claim that for any m satisfying $\frac{1}{\gamma} < m < \infty$, there is a date t_m such that

(a.14) $$R(t_m) > R(t_m - 1) > m$$

To establish this claim, note first that given any m, we can obviously select a subsequence $\{t_s\}$ such that (i) $m < R(t_s)$, (ii) $R(t_s) < R(t_{s+1})$ and (iii) $R(t_s) \to \infty$.

Next, note that given such a subsequence, we can also select a subsequence which satisfies besides (i) – (iii) the following additional property: (iv) $R(t_s) > R(t_s - 1)$. To show this, proceed iteratively as follows, starting with the original subsequence satisfying (i)–

[†]The opposite inequality cannot hold because the utility indicators have unbounded steepness at zero.

(iii). Suppose s' is the smallest s where (iv) is violated, and $R(t'_s - 1) \geq R(t_s)$. Then redefine t_s to equal $\hat{t}_s$, the smallest t that maximizes $R(t)$ in the set $\{t : t_{s-1} \leq t \leq t_s\}$, so that by construction (i) and (iv) holds at s. Next redefine t_{s+1} by selecting the next $s" \geq s+1$ with $R(t_{s"}) > R(\hat{t}_s)$, and setting t_{s+1} equal to $\hat{t}_{s+1}$, the smallest t that maximizes $R(t)$ in the set $\{t : \hat{t}_s \leq t \leq t_{s"}\}$. By construction we ensure (i), (ii) and (iv) in the redefined subsequence, and also (iii) because the value of R is raised in every corresponding element.

We are now in a position to prove the claim using the subsequence satisfying (i) –(iv). For if it were false, $R(t_s - 1) \leq m$ for all s. This would imply

$$R(t_s) - R(t_s - 1) \to \infty. \tag{a.15}$$

On the other hand, because the difference in productivity growth rates of the factor in the two sectors cannot exceed $\bar{\rho}$:

$$\frac{\pi_i(t_s)}{\pi_i(t_s - 1)} - \frac{\pi_j(t_s)}{\pi_j(t_s - 1)} \leq \bar{\rho},$$

This implies that for all s,

$$R(t_s) - R(t_s - 1) \leq \bar{\rho}\frac{\pi_i(t_s - 1)}{\pi_j(t_s)} = \bar{\rho}\left[\frac{\pi_i(t_s - 1)}{\pi_j(t_s - 1)}\right]\left[\frac{\pi_j(t_s - 1)}{\pi_j(t_s)}\right] \leq \bar{\rho}m,$$

given the contrary presumption that $R(t_s - 1) \leq m$ and given that $\pi_j(t_s - 1) \leq \pi_j(t_s)$. This contradicts (a.15) and establishes the claim in (a.14).

Returning to the main proof, let $a_i(t) \equiv \rho_i(t)\pi_i(t)$ be the absolute productivity advance in any sector i, and, for some fixed pair of sectors i and j, let $A(t) \equiv \sum_{k \neq i,j} a_k(t)$ be the aggregate productivity advance in all sectors apart from i and j. Consider first the case in which

$$\frac{1}{\pi_i(t_m)}\gamma[a_j(t_m) + A(t_m)] \geq \bar{\rho}. \tag{a.16}$$

Then $\rho_i(t_m) = \bar{\rho}$. Hence $a_j(t_m) + \gamma A(t_m) > \gamma[a_j(t_m) + A(t_m)] \geq \bar{\rho}\pi_i(t_m)$, while $\pi_i(t_m) > m\pi_j(t_m) > \pi_j(t_m)$ by (a.14). Therefore (a.16) implies that

$$\frac{1}{\pi_j(t_m)}[a_j(t_m) + \gamma A(t_m)] > \bar{\rho}$$

so that productivity growth rate in sector j must also equal $\bar{\rho}$, which contradicts (a.14). So (a.16) cannot hold, and therefore

$$\frac{\pi_i(t_m+1)}{\pi_i(t_m)} - 1 = \rho_i(t_m) + \frac{1}{\pi_i(t_m)}\gamma[a_j(t_m) + A(t_m)]. \tag{a.17}$$

Moreover, the rate of growth of productivity in sector j must be smaller than $\bar{\rho}$, otherwise (a.14) is contradicted again. So (a.17) also holds for sector j:

$$\frac{\pi_j(t_m+1)}{\pi_j(t_m)} - 1 = \rho_j(t_m) + \frac{1}{\pi_j(t_m)}\gamma[a_j(t_m) + A(t_m)], \tag{a.18}$$

and moreover, the right hand side of (a.18) must be smaller than the right hand side of (a.17); i.e.:

$$\rho_i(t_m) + \frac{1}{\pi_i(t_m)}\gamma[a_j(t_m) + A(t_m)] > \rho_j(t_m) + \frac{1}{\pi_j(t_m)}\gamma[a_i(t_m) + A(t_m)].$$

By (a.14) and $m > \frac{1}{\gamma} > 1$, and also recalling the definition of $a_i(t)$, this inequality implies:

$$\rho_i(t_m)\left[1 - \gamma\frac{\pi_i(t_m)}{\pi_j(t_m)}\right] > \rho_j(t_m)\left[1 - \gamma\frac{\pi_j(t_m)}{\pi_i(t_m)}\right] \tag{a.19}$$

But (a.19) cannot hold: its right hand side is positive (because $\frac{\pi_j(t_m)}{\pi_i(t_m)} < \frac{1}{m} < 1$) while its left hand side is negative (as $\gamma\frac{\pi_i(t_m)}{\pi_j(t_m)} > \gamma m > 1$). This contradiction proves the lemma.
■

Proof of Lemma 2. The proof relies on the following claim, which is parallel to Proposition 1. For any (θ_r, ν_r) with $\nu_r \geq \nu_r(0)$, there is a unique solution $p_r^*(\theta_r, \nu_r)$ to the equation

$$p_r = c_r\left(\frac{1}{\theta_r}, \frac{p_r}{\nu_r}\right), \tag{a.20}$$

and in any equilibrium, $p_r(t) \leq p_r^*(\theta_r(t), \nu_r(t))$ for all t.

No change in the proof of Proposition 1 is needed to prove this claim. We only note that (31) implies the singularity condition (17) for every $\nu_r \geq \nu_r(0)$.

With this claim in hand, we first establish (35) for $j = r$. Write $p_r^*(t) \equiv p_r^*(\theta_r(t), \nu_r(t))$. For any t, (a.20) tells us that

$$p_r^*(t) = c_r\left(\frac{1}{\theta_r(t)}, \frac{p_r^*(t)}{\nu_r(t)}\right).$$

Define $\eta(t) \equiv \nu_r(t)/p_r^*(t)\theta_r(t)$ for all t. Then, using the linear homogeneity of c_r, the above equality can be rewritten as

$$\nu_r(t) = c_r(\eta(t), 1), \tag{a.21}$$

for all t. It follows that

$$\nu_r(t+1) - \nu_r(t) = c_r(\eta(t+1), 1) - c_r(\eta(t+1), 1) \leq c_r^1(\eta(t), 1)[\eta(t+1) - \eta(t)],$$

where c_r^1 stands for the partial derivative of c_r with respect to its first argument, and the inequality above follows from the concavity of the unit cost function c_r. Using (a.21) again, it is easy to see that for every t,

$$\frac{\eta(t+1) - \eta(t)}{\eta(t)} \geq \frac{\nu_r(t+1) - \nu_r(t)}{\nu_r(t)}\left[\frac{c_r(\eta(t), 1)}{c_r^1(\eta(t), 1)\eta(t)}\right] \geq \frac{\nu_r(t+1) - \nu_r(t)}{\nu_r(t)},$$

where the last inequality is a consequence of the fact that c_r is concave. Notice also that (31) and (a.21) together guarantee that $\eta_r(0) > 0$. Putting all this information together, there exists $b_r > 0$ such that $\eta_r(t) \geq b_r\nu_r(t)$ for all t. Inverting this inequality, defining $B_r \equiv b_r^{-1} < \infty$, recalling the definition of $\eta(t)$, and noting that $p_r(t) \leq p_r^*(t)$ by the claim, we must conclude that

$$\theta_r(t)p_r(t) \leq \theta_r(t)p_r^*(t) \leq B_r < \infty \tag{a.22}$$

for all t, which establishes (35) for $j = r$. To complete the proof for all j, we recall part (b) of Lemma 1, which bounds the productivity ratios of each factor across sectors. That means we can replace $\theta_r(t)$ by $\theta_j(t)$ in the inequality (a.22) — adjusting B_r suitably to a new bound B_j — without jeopardizing boundedness. ■

Proof of Lemma 3. Pick some sector i satisfying Condition F (or F′). We know that in any competitive equilibrium and at any date t

(a.23) $$p_i(t)\theta_i(t)\frac{\partial f_i}{\partial[\theta_i k_i]}(t) = p_i(t)\theta_i(t)g_i'(e_i(t)) = 1,$$

where we recall the intensive form $g_i(e) = f_i(e,1)$ already defined. It follows that:

$$p_i(t)y_i(t) = \frac{y_i(t)}{\theta_i(t)g_i'(e_i(t))} = \left[\frac{y_i(t)}{\theta_i(t)k_i(t)g_i'(e_i(t))}\right]k_i(t).$$

Dividing above and below in this expression by the second input (human or robotic), we see that

(a.24) $$p_i(t)y_i(t) = \left[\frac{g_i(e_i(t))}{e_i(t)g_i'(e_i(t))}\right]k_i(t).$$

We claim next that under condition F, the sequence of input ratios $\{e_i(t)\}$ is bounded above and below by strictly positive, finite numbers. To this end, recall that

$$p_i(t) = c_i(\theta_i^{-1}(t), \lambda_i^*(t)),$$

where — following earlier notation — we let λ_i^* stand for the price of the second input in efficiency units. Therefore

(a.25) $$p_i(t)\theta_i(t) = c_i(1, \theta_i(t)\lambda_i^*(t)) \geq \epsilon$$

for some $\epsilon > 0$, by F. It follows from (a.23) that $e_i(t)$ must be bounded below by a strictly positive number.

Moreover, using (a.25) and recalling that $\lambda_i^*(t) \leq p_r(t)\nu_i^{-1}(t)$, we also see that

$$p_i(t)\theta_i(t) \leq c_i(1, \theta_i(t)p_r(t)\nu_i^{-1}(t)) \leq c_i(1, p_r(t)\theta_i(t)\nu_i^{-1}(0)) \leq c_i(1, B_i\nu_i^{-1}(0)),$$

where the very last inequality uses Lemma 2. Therefore $p_i(t)\theta_i(t)$ is bounded above in t. It follows from (a.23) that $e_i(t)$ must also be bounded above, establishing the claim.

It follows from this claim that the elasticity $e_i(t)g_i'(e_i(t))/g_i(e_i(t))$ is bounded below by some strictly positive number; call it a. (Notice that this is true by assumption if

Condition F′ is satisfied, without any need for the derived bounds on $e_i(t)$.) Using this information in (a.24), we must conclude that for all t,

(a.26) $$k_i(t) \geq ap_i(t)y_i(t).$$

Given our growth assumption, per-capita income goes to infinity. It follows from Condition E that the right hand side of (a.26) goes to infinity as well. So $k_i(t) \to \infty$, as claimed. ■