Appendix: Proof of propositions

Lemma 1.

$$\frac{d\beta}{dN}\frac{N}{\beta} > -1. \quad \text{If } N \geq 4, \qquad \frac{d\beta}{dN}\frac{N}{\beta} < 0.$$

Proof. Multiplying the cubic equation (8), $\rho\beta^3 + \frac{\tau_E}{N-1}\beta^2 + \frac{\sigma_z^2\tau_E}{N(N-1)}\rho\beta - \frac{(N-2)\sigma_z^2\tau_E^2}{N^2(N-1)^2} = 0$, by N^3, we obtain

$$\rho(N\beta)^3 + \frac{N\tau_E}{N-1}(N\beta)^2 + \frac{N\sigma_z^2\tau_E}{(N-1)}\rho N\beta - \frac{N(N-2)\sigma_z^2\tau_E^2}{(N-1)^2} = 0. \tag{1}$$

As N increases, the coefficients on the first three terms weakly decrease, while the last term increases. Therefore $N\beta$ should be increasing in N. Since $\beta > 0$, we obtain $\frac{d\beta}{dN}\frac{N}{\beta} > -1$.

Next, applying the implicit function theorem to $F(\beta, N) \equiv \rho\beta^3 + \frac{\tau_E}{N-1}\beta^2 + \frac{\sigma_z^2\tau_E}{N(N-1)}\rho\beta - \frac{(N-2)\sigma_z^2\tau_E^2}{N^2(N-1)^2} = 0$, we have

$$\frac{d\beta}{dN} = -\frac{\partial F/\partial N}{\partial F/\partial\beta} = -\left(\frac{\partial F}{\partial\beta}\right)^{-1}\left(-\frac{\tau_E}{(N-1)^2}\beta^2 - \frac{(2N-1)\sigma_z^2\tau_E\rho}{N^2(N-1)^2}\beta - \frac{(-3N^2+9N-4)\sigma_z^2\tau_E^2}{N^3(N-1)^3}\right), \tag{2}$$

where $\frac{\partial F}{\partial\beta} = 3\rho\beta^2 + \frac{2\tau_E}{N-1}\beta + \frac{\sigma_z^2\tau_E}{N(N-1)}\rho > 0$ is straightforwardly obtained because $\beta > 0$. Using $F(\beta, N) = 0$, the last term of $\partial F/\partial N$ is substituted out. The resulting expression is a polynomial of β without constant terms, and each coefficient of β is greater than zero if $N \geq 4$. Thus, we have $\partial F/\partial N > 0$ if $N \geq 4$, and $\frac{d\beta}{dN}\frac{N}{\beta} < 0$ can be obtained. ■

Regarding Lemma 1, the equation for β in (5) indicates why the increase of N results in a decrease of β. There are two channels through which N affects the behavior of traders; its effect on liquidity (which is represented by N in λ) and price informativeness (which is represented by τ_E and ρ). The liquidity channel is straightforward. Because the aggregate demand/supply function is linear in N, the increase of N makes the price inelastic with respect to the unit order flow: improvement of liquidity. In the imperfect competition model, when the market is more liquid (or the price is less elastic to the order flow), the traders become more eager to trade, i.e., β increases (this is implied by equation (5)).

The price informativeness channel is more complicated because it can increase as well as decrease β. If price becomes more informationally efficient and revealing, each trader becomes reluctant to trade on their signals, and β decreases. Because informational efficiency is increasing in N (i.e., $d\rho/dN > 0$), the larger N reduces β. On the other hand, if total available information increases in N, the precision of signal for traders τ_E increases accordingly. Then, trading on price information becomes less risky, traders become more aggressive, and β increases.

If we assume a monopolistic competition model, the second effect of informational channel is shut down, because total available information is fixed: $N\tau_E = \tau_I$. Therefore, information channel only reduces β, which can overcome the liquidity channel and an increase in N result in decreasing β.

Lemma 2.

$$\frac{d\varphi}{dN} > 0.$$

Proof. Differentiating the informational efficiency parameter $\varphi = \frac{(N-1)\beta^2}{(N-1)\beta^2+\sigma_z^2\tau_E}$ with respect to N, we have

$$\frac{d\varphi}{dN} = \left((N-1)\beta^2 + \frac{\sigma_z^2\tau_E}{N}\right)^{-2}\left\{\beta^2 + 2(N-1)\frac{d\beta}{dN}\beta - (N-1)\beta^2\left(2(N-1)\frac{d\beta}{dN} - \frac{\sigma_z^2\tau_E}{N^2}\right)\right\}. \tag{3}$$

The first parenthesis is clearly positive. The second parenthesis reduces to $(2-\frac{1}{N})\beta^2 + 2(N-1)\frac{d\beta}{dN}\beta$. After rearranging terms, we have

$$\frac{d\varphi}{dN} > 0 \quad\Leftrightarrow\quad \frac{d\beta/\beta}{dN/N} > -\frac{2N-1}{2(N-1)}.$$

Lemma 1 claims that this is satisfied in $N \geq 4$. ■

Lemma 3.

$$\frac{d\tau_I}{dN} < 0 \text{ if } \tau_E < \frac{(N-1)\sigma_z^2\rho}{2(N-2)}.$$

Proof. By definition (4), $\tau_I \equiv \tau_v + \tau_E/N + (N-1)\varphi\tau_E/N$. Differentiating with respect to N, we have

$$\frac{d\tau_I}{dN} = -\frac{\tau_E}{N^2} + \frac{d\varphi}{dN}(N-1)\frac{\tau_E}{N} + \varphi\frac{\tau_E}{N^2}.$$

Rearranging terms, we have

$$\frac{d\tau_I}{dN} < 0 \quad\Leftrightarrow\quad \frac{d\varphi}{dN} < \frac{1-\varphi}{N(N-1)}.$$

Substituting out $d\varphi/dN$ with the equation (16), this is rewritten:

$$\frac{d\tau_I}{dN} < 0 \quad\Leftrightarrow\quad \frac{d\varphi}{dN} < \frac{1-\varphi}{N(N-1)} \quad\Leftrightarrow\quad \frac{d\beta/\beta}{dN/N} < -1 + \frac{\sigma_z^2\tau_E}{2N(N-1)^2\beta^2}.$$

We have the expression for $d\beta/dN$ with the equation (15). Rearranging it yields

$$\frac{d\beta/\beta}{dN/N} = -\frac{-\frac{N\tau_E}{(N-1)^2}\beta^2 - \frac{(2N-1)\sigma_z^2\tau_E\rho}{N(N-1)^2}\beta - \frac{(-3N^2+9N-4)\sigma_z^2\tau_E^2}{N^2(N-1)^3}}{-\frac{\tau_E}{(N-1)}\beta^2 - \frac{2\sigma_z^2\tau_E\rho}{N(N-1)}\beta + \frac{3(N-2)\sigma_z^2\tau_E^2}{N^2(N-1)^2}}.$$

Here, the denominator is obtained by substituting $3\rho\beta^3$ out by $F(\beta, N) = 0$. Using this expression, after some calculations, we have

$$\frac{d\beta/\beta}{dN/N} < -1 + \frac{\sigma_z^2 \tau_E}{2N(N-1)^2\beta^2}$$

$$\Leftrightarrow \ \frac{\tau_E}{(N-1)}\beta^3 + \left(1 - \frac{3}{2}\right)\frac{\sigma_z^2 \tau_E \rho}{N(N-1)}\beta^2 + \left(\frac{2}{N} - 1\right)\frac{\sigma_z^2 \tau_E^2}{N(N-1)^2}\beta - \frac{\rho \sigma_z^4 \tau_E^2}{2N^2(N-1)^2} < 0.$$

Again, we substitute out the last term by using $F(\beta, N) = 0$, and make sure each coefficient is negative. In the resulting expression, the coefficients on β^2 and β are clearly negative. The coefficients on β^3 is $\frac{\tau_E}{(N-1)} - \frac{\sigma_z^2 \rho}{2(N-2)}$, which can be negative if $\tau_E < \frac{(N-1)\sigma_z^2 \rho}{2(N-2)}$. ∎

Corollary 1.

$$\frac{dR^2}{dN} < 0 \ \text{ and } \ \frac{dV}{dN} > 0 \ \text{ if } \ \ \tau_E < \frac{(N-1)\sigma_z^2\rho}{2(N-2)}.$$

Proof. $R^2 \equiv 1 - [illegible]/[illegible]$ is a definition. The sign of dR^2/dN coincides with $d[illegible]/dN$ because $\frac{dR^2}{dN} = \frac{[illegible]}{[illegible]^2}\frac{d[illegible]}{dN}$. $V \equiv (1 + [illegible]/[illegible])(R^2)^2$ and $dV/dN > 0$ immediately follow. ∎

Corollary 2.

$$\frac{d\lambda}{dN} < 0 \quad \text{if} \quad \tau_E < \frac{(N-1)\sigma_z^2\rho}{2(N-2)}.$$

Proof. As equation (7) indicates, the price impact is $\lambda = \frac{1}{[illegible]}\left(1 - \frac{[illegible]}{[illegible]}\right)$. Lemma 1 suggests that $1/[illegible]$ is decreasing in N, and Corollary 1 suggests $1 - [illegible]/[illegible]$ is also decreasing in N. Thus λ is decreasing in N. ∎

Lemma 4.

$$\frac{d\beta}{d\tau_E} > 0.$$

Proof. Applying the implicit function theorem to $F(\beta, \tau_E) \equiv \rho\beta^3 + \frac{\tau_E}{N-1}\beta^2 + \frac{\sigma_z^2\tau_E}{N(N-1)}\rho\beta - \frac{(N-2)\sigma_z^2\tau_E^2}{N^2(N-1)^2} = 0$, we have

$$\frac{d\beta}{d\tau_E} = -\frac{\partial F/\partial \tau_E}{\partial F/\partial \beta} = -\left(\frac{\partial F}{\partial \beta}\right)^{-1}\left(\frac{1}{(N-1)}\beta^2 + \frac{\sigma_z^2\rho}{N(N-1)}\beta - \frac{2(N-2)\sigma_z^2\tau_E}{N^2(N-1)^2}\right). \tag{4}$$

We can show $\partial F/\partial \beta > 0$ like Lemma 1. The last term is substitute out by $F(\beta, \tau_E) = 0$, and we have

$$\frac{\partial F}{\partial \tau_E} = -\frac{2\rho\beta^3}{\tau_E} + \left(\frac{1}{N-1} - \frac{2}{N-1}\right)\beta^2 + \left(\frac{1}{N(N-1)} - \frac{2}{N(N-1)}\right)\sigma_z^2\rho\beta < 0,$$

because each term in parentheses is negative and [illegible] > 0. Combining these, we obtain [illegible]$/$[illegible] > 0. ■

Lemma 5.

$$\frac{d\varphi}{d\tau_E} > 0.$$

Proof. By definition (4), $\varphi \equiv \frac{(N-1)\beta^2}{(N-1)\beta^2+\sigma_z^2\tau_E/N}$. Differentiating with respect to by τ_E and rearranging terms, we have

$$\frac{d\varphi}{d\tau_E} = \left((N-1)\beta^2 + \frac{\sigma_z^2\tau_E}{N}\right)^{-2}\left\{2(N-1)\frac{d\beta}{d\tau_E}\beta\frac{\sigma_z^2\tau_E}{N} - (N-1)\beta^2\frac{\sigma_z^2}{N}\right\}.$$

The denominator is clearly positive, and it is enough to show the numerator is positive as well. Rearranging terms, we can show that the numerator is positive if and only if

$$\frac{d\beta}{d\tau_E}\frac{\tau_E}{\beta} > \frac{1}{2}.$$

In the LHS, substituting out $d\beta/d\tau_E$ by (17), we have

$$\frac{d\beta}{d\tau_E}\frac{\tau_E}{\beta} = -\frac{\frac{\tau_E}{(N-1)}\beta^2 + \frac{\sigma_z^2\tau_E\rho}{N(N-1)}\beta - \frac{2(N-2)\sigma_z^2\tau_E^2}{N^2(N-1)^2}}{-\frac{\tau_E}{(N-1)}\beta^2 - \frac{2\sigma_z^2\tau_E\rho}{N(N-1)}\beta + \frac{3(N-2)\sigma_z^2\tau_E^2}{N^2(N-1)^2}}. \tag{5}$$

Note that the denominator is positive. Rearranging terms, we have

$$\frac{d\beta}{d\tau_E}\frac{\tau_E}{\beta} > \frac{1}{2} \iff \frac{-\tau_E}{(N-1)}\beta^2 + \frac{(N-2)\sigma_z^2\tau_E^2}{N^2(N-1)^2} > 0 \iff \rho\beta^3 + \frac{\sigma_z^2\tau_E\rho}{N(N-1)}\beta > 0.$$

Last equivalence holds from [illegible] $= 0$. ■

Note that, evaluating $\frac{d\varphi}{d\tau_E}$, we have $\frac{d\varphi}{d\tau_E}\frac{\tau_E}{\varphi} = (1-\varphi)\left(2\frac{d\beta}{d\tau_E}\frac{\tau_E}{\beta} - 1\right)$. We can also argue the limiting case of $\tau_E \to \infty$ and $N \to \infty$. From equation 14), with $N \to \infty$, we have,

$$\rho(N\beta)^3 + \tau_E(N\beta)^2 + \rho\sigma_z^2\tau_E N\beta - \sigma_z^2\tau_E^2 = 0.$$

Here $N\beta$ should be finite to satisfy the equality. Taking the limit for τ_E, we obtain $(N\beta)^2 = \sigma_z^2\tau_E$. Plugging this into the definition of φ, we find that $\lim_{\tau_E\to\infty, N\to\infty} \varphi = 1/2$.[1]

Lemma 6.

$$\frac{d\tau_I}{d\tau_E} > 0.$$

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[1] The result $0 < \varphi < \frac{1}{2}$ is also stated in Kyle (1989).

Proof. Differentiating $□_□ \equiv □_□ + \frac{□_□}{□} + \frac{(□-1)□_□}{□}□(□_□)$ with respect to $□_□$, we have $\frac{□□_□}{□□_□} = \frac{1}{□} + \frac{(□-1)}{□}□(□_□) + \frac{(□-1)□_□}{□}\frac{□□}{□□_□}$. Applying Lemma 5, with $□ > 0$ by definition, we have the desired result. ■

Corollary 3.

$$\frac{dR^2}{d\tau_E} > 0.$$

Proof. Differentiating of $□^2 = \left(1 - \frac{□_□}{□_□(□_□)}\right)$ by $□_□$, we have $\frac{□□^2}{□□_□} = \frac{□_□}{□_□^2}\frac{□□_□}{□□_□}$. Applying Lemma 6, we have the desired result.

Lemma 7.

$$For\ any\ finite\ exogenous\ parameters, there\ is\ sufficiently\ large\ \tau_E\ that\ satisfies \frac{d\lambda}{d\tau_E} < 0.$$

$$Also, for\ any\ finite\ \tau_E, there\ is\ sufficiently\ large\ \tau_v\ that\ satisfies\ \frac{d\lambda}{d\tau_E} > 0.$$

Proof. Since $□ = □^2/□□$, taking a derivative of $\log □$ with respect to $□_□$, we have $\frac{□□}{□□_□}\frac{□_□}{□} = \frac{□□^2}{□□_□}\frac{□_□}{□^2} - \frac{□□}{□□_□}\frac{□_□}{□}$. Since both $□$ and $□_□$ are positive, we examine the sign of $\frac{□□^2}{□□_□}\frac{□_□}{□^2} - \frac{□□}{□□_□}\frac{□_□}{□}$ to explore the sign of $\frac{□□}{□□_□}$.

Because of the definition of $□_□$ and $□$ is bounded, as $□_□$ goes to infinity, $□_□/□_□$ approach to 0 and $□^2 \to 1$. After some calculation, ignoring exogenous constants, we have $\frac{□□^2}{□□_□}\frac{□_□}{□^2} = \frac{□_□}{□_□^2}\frac{□□_□}{□□_□}\frac{□_□}{□^2} \sim \frac{1}{□_□^2}\left(□_□ + \frac{□□}{□□_□}□_□^2\right)$, which goes to zero as $□_□$ goes to infinity because $\frac{□□}{□□_□} \to 0$ ($□$ is increasing but bounded). Since $\frac{□□}{□□_□}\frac{□_□}{□} > 1/2$, $\frac{□□}{□□_□}\frac{□_□}{□}$ becomes negative if $□_□$ is sufficiently large. The cutoff value depends on other exogenous parameters.

Next, we show the second statement. Fist, we show that $\frac{□□}{□□_□}\frac{□_□}{□} < 1$. From equation (18), a direct calculation leads $\frac{□□}{□□_□}\frac{□_□}{□} < 1 \Leftrightarrow □□ - \frac{□-2}{□(□-1)}□_□ < 0$. Plugging $□^* = \frac{□-2}{□□(□-1)}□_□$ into the cubic equation (8), we find that $□(□, □^*) > 0$. Since $□(□, □)$ is an increasing function of $□$, $□$ that satisfies $□(□, □) = 0$ is less than $□^*$. Thus, we can show that $□□ - \frac{□-2}{□(□-1)}□_□ < 0$. Second, we show that we can find $□_□$ that satisfies $\frac{□□^2}{□□_□}\frac{□_□}{□^2} > \frac{□□}{□□_□}\frac{□_□}{□}$. Expanding $\frac{□□^2}{□□_□}\frac{□_□}{□^2}$, we have

$$\frac{dR^2}{d\tau_E}\frac{\tau_E}{R^2} = \frac{\tau_v}{\tau_I}\left(1 + (1-\varphi)\left(2\frac{d\beta}{d\tau_E}\frac{\tau_E}{\beta} - 1\right)\varphi\frac{N-1}{1+\varphi(N-1)}\right) \tag{6}$$

Thus, collecting terms on $□_□$ for $\frac{□□^2}{□□_□}\frac{□_□}{□^2} > \frac{□□}{□□_□}\frac{□_□}{□}$, we can find the condition for $□_□$. Note that $□, □, \frac{□□}{□□_□}\frac{□_□}{□}$ are not a function of $□_□$ and $\frac{1}{2} < \frac{□□}{□□_□}\frac{□_□}{□} < 1$ and $0 < □ < \frac{1}{2}$ ensure the existence of such $□_□$. For a limiting case, the condition for $□_□$ reduces to $□_□ > \frac{□}{1-□}□_□ \to □_□$ when $□ \to \infty$. ■

Note that we can also show that $\frac{□□}{□□_□}\frac{□_□}{□}$ approaches to zero when τ_E goes to zero (from equation (8), (18)).

Lemma 8.

$$\frac{dV}{d\tau_E} > 0.$$

Proof. Note that $V = \left(1+\frac{□_□}{□_□}\right)\left(□^2\right)^2 = \left(1+\frac{□_□}{□_□}\right)\left(1-\frac{□_□}{□_□}\right)^2$. Substituting out $□_□$, we obtain

$$V = \left(1+\frac{\tau_v}{\tau_E}\right)\left(\frac{1+(N-1)\varphi}{1+(N-1)\varphi+N\tau_v/\tau_E}\right)^2.$$

Since $V > 0$, for obtaining $\frac{dV}{d\tau_E} > 0$, it is enough to show $\frac{d\ln V}{d\tau_E} > 0$;

$$\frac{d\ln V}{d\tau_E} = \frac{d\varphi}{d\tau_E}\left(\frac{2(N-1)}{1+(N-1)\varphi} - \frac{2(N-1)}{1+(N-1)\varphi+N\tau_v/\tau_E}\right) - \frac{N\tau_v}{\tau_E^2}\left(\frac{1}{N+N\tau_v/\tau_E} - \frac{2}{1+(N-1)\varphi+N\tau_v/\tau_E}\right).$$

From Lemma 5 we have $d\varphi/d\tau_E > 0$, and the first parenthesis is positive because $\varphi > 0$. Also, we can show the second parenthesis is negative. This results in $d\ln V/d\tau_E > 0$. ■